

LIE SUPERGROUPS vs. SUPER HARISH-CHANDRA PAIRS: A NEW EQUIVALENCE

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Abstract

¹ This paper presents a new contribution to the study of Lie supergroups — of real smooth, real analytic or complex holomorphic type — by means of super Harish-Chandra pairs. Namely, one knows that there exists a natural functor Φ from Lie supergroups to super Harish-Chandra pairs: a functor going backwards, that associates a Lie supergroup with each super Harish-Chandra pair, yielding an equivalence of categories, was found by Koszul [17]. This result was later extended by other authors, to different levels of generality, but always elaborating on Koszul’s original idea.

In this paper, I provide a new backwards equivalence, i.e. a different functor Ψ that constructs a Lie supergroup out of a given super Harish-Chandra pair, so that any Lie supergroup is recovered from its naturally associated super Harish-Chandra pair; more precisely, Ψ and Φ are quasi-inverse to each other.

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1 Introduction

The study of supergroups is a chapter of supergeometry, i.e. geometry in a \mathbb{Z}_2 -graded sense. In particular, the relevant structure sheaves of (commutative) algebras sitting on top of the topological spaces one works with are replaced with sheaves of (commutative) *superalgebras*.

When dealing with *differential* supergeometry, our “superspaces” are supermanifolds, that are real smooth, real analytic or complex holomorphic (depending on the context): any such supermanifold can be considered as a *classical* (i.e. non-super) manifold – of the appropriate type — endowed with a suitable sheaf of commutative superalgebras. The supergroups in this context are then *Lie* supergroups, of smooth, analytic or holomorphic type according to the chosen setup.

For every Lie supergroup G there exists a special pair of objects, say (G_0, \mathfrak{g}) , that is naturally associated with it: G_0 is the *classical Lie group underlying* G — roughly given by “killing the odd part” of the structure sheaf on G — while $\mathfrak{g} = \text{Lie}(G)$ is the *tangent Lie superalgebra of* G , and these two objects are “compatible” in a natural sense. More in general, any similar pair (K_+, \mathfrak{k}) made by a Lie group K and a Lie superalgebra \mathfrak{k} obeying the same compatibility constraints is called “super Harish-Chandra pair” (a terminology first found in [8]), or just “ \mathbf{sHCp} ” for short: in fact, this notion is tailored in such a way that mapping $G \mapsto (G_0, \mathfrak{g})$ yields a functor, call it Φ , from the category of Lie supergroups — either smooth, analytic or holomorphic — to the category of super Harish-Chandra pairs — of smooth, analytic or holomorphic type respectively.

The key question then is: can one recover a Lie supergroup out of its associated \mathbf{sHCp} ? Or, even more precisely: does there exist any functor Ψ from \mathbf{sHCp} ’s to Lie supergroups which be a quasi-inverse for Φ , so that the two categories be equivalent? And if the answer is positive, how much explicit such a functor is?

A first answer to this question was given by Kostant and by Koszul in the real smooth case (see [16] and [17]), with equivalent methods, providing an explicit quasi-inverse for Φ . Later on, Vishnyakova (see [24]) fixed the complex holomorphic case, and her proof works for the real analytic case as well. More recently Carmeli and Fioresi (see [7]) raised and solved the same problem in the setup of *algebraic supergeometry*, i.e. for algebraic supergroups (and corresponding \mathbf{sHCp} ’s), over a ground ring \mathbb{k} that is an algebraically closed field of characteristic zero. This was improved by Masuoka (in [20]), who only required that \mathbb{k} be a field whose characteristic is not 2; and later on (see [21]), Masuoka and Shibata further extended this result up to work on every commutative ring. It is worth remarking that all these (increasingly deeper) results were, in the end, further improvements of the original idea by Koszul in [17] (while Kostant’s method was a slight variation of that): indeed, Koszul defines a Lie supergroup out of a \mathbf{sHCp} (K_+, \mathfrak{k}) as a super-ringed space,

just defining the “proper” sheaf of (commutative) superalgebras onto K_+ by means of the Lie superalgebra \mathfrak{k} ; the subsequent authors above then conveniently re-worked this same recipe.

In this paper I present a new method to solve that problem, i.e. I provide a different, more concrete functor Ψ from super Harish-Chandra pairs to Lie supergroups that is quasi-inverse to Φ . The starting idea is to follow a different approach to supergeometry, à la Grothendieck: namely instead of thinking of supermanifolds as being super-ringed manifolds (i.e. classical manifolds endowed with a sheaf of commutative superalgebras), one studies (or directly defines) them through their “functor of points”. Thus, if M is a supermanifold, then for each commutative superalgebra A one has the manifold $M(A)$ of A -points of M ; in fact, in order to recover the full supermanifold M one can restrict this functor to a smaller category, namely that of *Weil superalgebras* — roughly, those whose even part is made of a copy of our ground field (namely \mathbb{R} or \mathbb{C}) plus a finite-dimensional nilpotent ideal. Conversely, functors from Weil superalgebras to manifolds enjoying some additional properties do correspond to Lie supergroups (i.e., they are the functor of points of some Lie supergroup): so one can directly call “Lie supergroup” any such special functor. This functorial point of view has proved being definitely fit to unify several different approaches to supergeometry (see [2]) and also to be “the” right one to treat infinite-dimensional supermanifolds too (see [1]). For a broader discussion of the interplay between different approaches to supergeometry the reader may refer to classical sources as [3], [8], [18], [25] or more recent ones like [2], [4], [6], [22], [23].

Now, if we want a functor Ψ from sHCp’s to Lie supergroups, we need a Lie supergroup $G_{\mathcal{P}}$ for each sHCp \mathcal{P} ; using the functorial point of view, in order to have $G_{\mathcal{P}}$ as a functor we need a Lie group $G_{\mathcal{P}}(A)$ for each Weil superalgebra A , and their definition must be natural in A : moreover, one still has to show that the resulting functor have those additional properties that make it into a Lie supergroup. Finally, all this should aim to find a Ψ that is quasi-inverse to Φ — and this fixes ultimate bounds to the construction we aim to.

Bearing all this in mind, the construction that I present goes as follows. Given a super Harish-Chandra pair $\mathcal{P} = (G_+, \mathfrak{g})$, for each Weil superalgebra, say A , I define a group $G_{\mathcal{P}}(A)$ abstractly, by generators and relations: this definition is uniform with respect to A , and natural (in A), hence it yields a functor from Weil algebras to (abstract) groups, call it $G_{\mathcal{P}}$. As key step in the work, one proves that $G_{\mathcal{P}}$ has a special structure — called “*global splitting*” — namely it is the direct product of G_+ times a totally odd affine superspace (isomorphic to \mathfrak{g}_1 , the odd part of \mathfrak{g}): as both these are supermanifolds, it turns out that $G_{\mathcal{P}}$ itself is a supermanifold as well, and in fact it is a Lie supergroup because (as a functor) it is group-valued too. With yet another step, one shows that the construction of $G_{\mathcal{P}}$ is natural in \mathcal{P} , hence it yields a functor Ψ from sHCp’s to Lie supergroups: this is our candidate to be a quasi-inverse to Φ .

It is immediate to check that the composition $\Phi \circ \Psi$ is isomorphic to the identity functor onto super Harish-Chandra pairs — that is, $\Phi(G_{\mathcal{P}}) \cong \mathcal{P}$ for every super Harish-Chandra pair \mathcal{P} . On the other hand, proving that $\Psi \circ \Phi$ is isomorphic to the identity functor onto Lie supergroups — that is, $\Psi(\Phi(G)) \cong G$ for every Lie supergroup G — is much more demanding. In fact, in order to get this we need to recall that every Lie supergroup G has a “global splitting” on its own, just like $\Psi(\Phi(G))$ has: this implies that G and $\Psi(\Phi(G))$ share the same structure, in that both are the direct product of G_0 and of a copy of $\mathfrak{g}_1 = (\text{Lie}(G))_1$. Indeed, the fact that a “global splitting theorem” for Lie supergroups does hold true seem to be (more or less) known among specialists; however, we need it stated in a genuine geometrical form, while it is usually given in sheaf-theoretic terms (as far as I know), so in the end we have to work it out explicitly in order to apply it to our present problem: in turn, this requires quite a bit of work on its own.

Two last words are still in order:

(a) The same recipe given here for the functor Ψ was first presented in [15] — and the key idea was also in [10], [11], [12], [13] and [14] — to solve the same problem in the context of

algebraic supergeometry; in that paper, I loosely mentioned, that the given recipe would apply to the differential setup, i.e. to Lie supergroups and their sHCp's, if one treated it with the functorial language. Nevertheless, I later realized that such a switch from the algebraic to the differential context is not that trivial, also because the functorial approach in differential supergeometry is not so widely known (or applied): this is why, eventually, I decided to write down the present paper.

(b) Although in the present work we deal with Lie supergroups (and sHCp's) of finite dimension, our construction of the functor Ψ is perfectly fit for the *infinite dimensional* case as well — still following the most updated approach via the functorial language, as in [1]. Clearly, this requires some fine-tuning technicalities, which goes beyond the scopes of the present work, so we do not fulfill that task; however, the core idea (and strategy) to follow is already displayed hereafter.

Finally, the paper is organized as follows. In section 2 we establish language and notations, in particular all what is needed to deal with the functorial approach to Lie supergroups. Then section 3 presents the notion of super Harish-Chandra pair and the natural functor Φ from Lie supergroups to super Harish-Chandra pairs. Section 4 presents structure results for Lie supergroups, in particular about “global splittings”: more or less, these results are known (or should be known), but we could not find them in literature so we write them down ourselves in the form we need them to be.

The core of the paper is in sections 5 and 6. In section 5 we introduce the definition of the functor Ψ , and we prove key structure results for the Lie supergroups $G_{\mathcal{P}} := \Psi(\mathcal{P})$; in fact, it is worth stressing that the very definition of $G_{\mathcal{P}}$ and the results about its structure are somehow “prescribed” by the (similar) structure results for Lie supergroups in general that are presented in section 4. Then in section 6 we prove, using the structure results of sections 4 and 5 (in particular, the “global splitting theorems”), that the functor Ψ is indeed a quasi-inverse to Φ , as expected.

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2 Preliminaries

In all this work, \mathbb{K} will denote the field \mathbb{R} or \mathbb{C} of real or complex numbers, respectively, following the context. All modules (i.e., vector spaces), algebras etc. will be considered over \mathbb{K} .

2.1 Supermodules and superalgebras

2.1.1. Basic algebraic superobjects. We call \mathbb{K} -*supermodule*, or \mathbb{K} -*super vector space*, any \mathbb{K} -module V endowed with a \mathbb{Z}_2 -grading, say $V = V_0 \oplus V_1$, where $\mathbb{Z}_2 = \{0, 1\}$ is the group with two elements. The \mathbb{K} -submodule V_0 and its elements are called *even*, while V_1 and its elements *odd*. By $|x|$ or $p(x)$ ($\in \mathbb{Z}_2$) we denote the *parity* of any homogeneous element, which is defined by the condition $x \in V_{|x|}$.

We call \mathbb{K} -*superalgebra* any associative, unital \mathbb{K} -algebra A which is \mathbb{Z}_2 -graded (as a \mathbb{K} -algebra): so A has a \mathbb{Z}_2 -splitting $A = A_0 \oplus A_1$, and $A_{\mathbf{a}} A_{\mathbf{b}} \subseteq A_{\mathbf{a}+\mathbf{b}}$. All \mathbb{K} -superalgebras form a category, whose morphisms are those of unital \mathbb{K} -algebras that preserve (also) the \mathbb{Z}_2 -grading.

A superalgebra A is said to be *commutative* — in the “super sense” — iff $xy = (-1)^{|x||y|}yx$ for all homogeneous $x, y \in A$ and $z^2 = 0$ for all odd $z \in A_1$.

We denote by **(salg)** the category of commutative \mathbb{K} -superalgebras; if necessary, we shall stress the rôle of \mathbb{K} by writing **(salg) $_{\mathbb{K}}$** . Moreover, we shall denote by **(alg)** — or **(alg) $_{\mathbb{K}}$** , sometimes — the category of (associative, unital) commutative \mathbb{K} -algebras, and by **(mod) $_{\mathbb{K}}$** that of \mathbb{K} -modules.

For $A \in (\mathbf{salg})$, $n \in \mathbb{N}$, we call $A_1^{[n]}$ the A_0 -submodule of A spanned by all products $\vartheta_1 \cdots \vartheta_n$ with $\vartheta_i \in A_1$ for all i . We need also the following constructions: if $J_A := (A_1)$ is the ideal of A generated by A_1 , then $J_A = A_1^{[2]} \oplus A_1$, and $\overline{A} := A/J_A$ is a commutative superalgebra which is *totally even*, i.e. $\overline{A} \in (\mathbf{alg})$; also, there is an obvious isomorphism $\overline{A} := A/(A_1) \cong A_0/A_1^{[2]}$.

Finally, the constructions of A_0 , of $A_1^{(n)}$ and of \overline{A} all are functorial in A .

2.1.2. Weil superalgebras. We now introduce a special class of commutative superalgebras, the *Weil superalgebras*, or “*super Weil algebras*”, to be used later on (cf. [2] and references therein).

Definition 2.1.3. We call *Weil superalgebra* any finite-dimensional commutative \mathbb{K} -superalgebra A such that $A = \mathbb{K} \oplus \mathring{A}$ where \mathbb{K} is even and $\mathring{A} = \mathring{A}_0 \oplus \mathring{A}_1$ is a \mathbb{Z}_2 -graded nilpotent ideal (the *nilradical* of A). By construction, every Weil superalgebra A is automatically endowed with the canonical (super)algebra morphisms $p_A : A \longrightarrow \mathbb{K}$ and $u_A : \mathbb{K} \longrightarrow A$ associated with the direct sum splitting $A = \mathbb{K} \oplus \mathring{A}$; in particular $p_A \circ u_A = id_{\mathbb{K}}$, so that p_A is surjective and u_A is injective. Weil superalgebras over \mathbb{K} form a full subcategory of $(\mathbf{salg})_{\mathbb{K}}$, denoted $(\mathbf{Wsalg})_{\mathbb{K}}$ or $(\mathbf{Wsalg})_{\mathbb{K}}$. \diamond

A special class of Weil superalgebras is given by *Grassmann algebras*: namely, for any $n \in \mathbb{N}$ by *Grassmann algebra in n variables over \mathbb{K}* we mean the polynomial \mathbb{K} -algebra $\Lambda_n := \mathbb{K}[\xi_1, \dots, \xi_n]$ in n mutually anticommuting indeterminates ξ_1, \dots, ξ_n . Giving degree $\mathbf{1}$ to every variable ξ_i all these Grassmann algebras turn into commutative \mathbb{K} -superalgebras; we denote by $(\mathbf{Grass})_{\mathbb{K}}$, or just (\mathbf{Grass}) , the full subcategory of $(\mathbf{Wsalg})_{\mathbb{K}}$ whose objects are isomorphic to some Λ_n ($n \in \mathbb{N}$).

2.2 Lie superalgebras

The infinitesimal counterpart of Lie supergroups is given by the notion of Lie superalgebras. We shall see their link later on, while now we fix that notion, as well as its “functorial formulation”.

Definition 2.2.1. Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a \mathbb{K} -supermodule. We say that \mathfrak{g} is a *Lie superalgebra* if we have a (*Lie super*)*bracket* $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$, $(x, y) \mapsto [x, y]$, which is \mathbb{K} -bilinear, \mathbb{Z}_2 -graded and satisfies the following properties (for all $x, y, z \in \mathfrak{g}_0 \cup \mathfrak{g}_1$):

- (a) $[x, y] + (-1)^{|x||y|}[y, x] = 0$ (*anti-symmetry*);
- (b) $(-1)^{|x||z|}[x, [y, z]] + (-1)^{|y||x|}[y, [z, x]] + (-1)^{|z||y|}[z, [x, y]] = 0$ (*Jacobi identity*).

In this situation, as a matter of notation, we shall write

$$(c) \quad Y^{\langle 2 \rangle} := 2^{-1} [Y, Y] \quad (\in \mathfrak{g}_0) \quad \text{for all } Y \in \mathfrak{g}_1.$$

All Lie \mathbb{K} -superalgebras form a category, denoted $(\mathbf{sLie})_{\mathbb{K}}$ or just (\mathbf{sLie}) , whose morphisms are the \mathbb{K} -linear, graded maps that preserve the bracket. \diamond

Note that if \mathfrak{g} is a Lie \mathbb{K} -superalgebra, then its even part \mathfrak{g}_0 is automatically a Lie \mathbb{K} -algebra.

Example 2.2.2. Let $V = V_0 \oplus V_1$ be a \mathbb{K} -supermodule, and consider $End(V)$, the endomorphisms of V as an ordinary K -module. This is in turn a \mathbb{K} -supermodule, $End(V) = End(V)_0 \oplus End(V)_1$, where $End(V)_0$ are the morphisms which preserve the parity, and $End(V)_1$ those which reverse it. If V has finite dimension and we choose a basis for V of homogeneous elements (writing first the even ones), then $End(V)_0$ is the set of all diagonal block matrices, while $End(V)_1$ is

the set of all off-diagonal block matrices. Thus $End(V)$ is a Lie \mathbb{K} -superalgebra with bracket $[A, B] := AB - (-1)^{|A||B|}BA$ for homogeneous $A, B \in End(V)$; and then $Y^{\langle 2 \rangle} = Y^2$ for odd Y .

The standard example is $V := \mathbb{K}^p \oplus \mathbb{K}^q$, with $V_0 := \mathbb{K}^p$ and $V_1 := \mathbb{K}^q$. In this case we also write $End(\mathbb{K}^{m|n}) := End(V)$ or $\mathfrak{gl}_{p|q} := End(V)$. \blacklozenge

2.2.3. Functorial presentation of Lie superalgebras. Let $(\mathbf{Wsalg})_{\mathbb{K}}$ be the category of Weil \mathbb{K} -superalgebras (see §2.1) and denote by $(\mathbf{Lie})_{\mathbb{K}}$ the category of Lie \mathbb{K} -algebras.

Every Lie \mathbb{K} -superalgebra $\mathfrak{g} \in (\mathbf{sLie})_{\mathbb{K}}$ yields a functor

$$\mathcal{L}_{\mathfrak{g}} : (\mathbf{Wsalg})_{\mathbb{K}} \longrightarrow (\mathbf{Lie})_{\mathbb{K}}, \quad A \mapsto \mathcal{L}_{\mathfrak{g}}(A) := (A \otimes \mathfrak{g})_0 = (A_0 \otimes \mathfrak{g}_0) \oplus (A_1 \otimes \mathfrak{g}_1)$$

Indeed, $A \otimes \mathfrak{g}$ is a Lie superalgebra (in a suitable, more general sense, on the Weil \mathbb{K} -superalgebra A) on its own, with Lie bracket $[a \otimes X, a' \otimes X'] := (-1)^{|X||a'|} a a' \otimes [X, X']$, given by the so-called “sign rules”; now $\mathcal{L}_{\mathfrak{g}}(A)$ is the even part of the Lie superalgebra $A \otimes \mathfrak{g}$, hence it is a Lie algebra on its own (see [6] for details). In particular, all this applies to $\mathfrak{g} := End(V)$.

More in general, the following holds. Every \mathbb{K} -supermodule $\mathfrak{m} = \mathfrak{m}_0 \oplus \mathfrak{m}_1$ defines a functor

$$\mathcal{L}_{\mathfrak{m}} : (\mathbf{Wsalg})_{\mathbb{K}} \longrightarrow (\mathbf{mod})_{\mathbb{K}}, \quad A \mapsto \mathcal{L}_{\mathfrak{m}}(A) := (A \otimes_{\mathbb{K}} \mathfrak{m})_0 = (A_0 \otimes \mathfrak{m}_0) \oplus (A_1 \otimes \mathfrak{m}_1)$$

then \mathfrak{m} is a Lie \mathbb{K} -superalgebra $\iff \mathcal{L}_{\mathfrak{m}}$ takes values in $(\mathbf{Lie})_{\mathbb{K}}$.

In fact, this holds true also if we replace $(\mathbf{Wsalg})_{\mathbb{K}}$ with its (smaller) subcategory $(\mathbf{Grass})_{\mathbb{K}}$.

This “functorial presentation” of Lie superalgebras can be adapted to representations too. Indeed, let V be a \mathfrak{g} -module, with representation map $\phi : \mathfrak{g} \longrightarrow End(V)$ — a Lie superalgebra morphism. Scalar extension induces a morphism $id_A \otimes \phi : A \otimes \mathfrak{g} \longrightarrow A \otimes End(V)$ for each $A \in (\mathbf{Wsalg})_{\mathbb{K}}$, whose restriction to the even part gives a morphism $(A \otimes \mathfrak{g})_0 \longrightarrow (A \otimes End(V))_0$, that is a morphism $\mathcal{L}_{\mathfrak{g}}(A) \longrightarrow \mathcal{L}_{End(V)}(A)$ in $(\mathbf{Lie})_{\mathbb{K}}$. The whole construction is natural in A , hence it induces a natural transformation of functors $\mathcal{L}_{\mathfrak{g}} \longrightarrow \mathcal{L}_{End(V)}$.

The same considerations apply as well if $(\mathbf{Wsalg})_{\mathbb{K}}$ is replaced with $(\mathbf{Grass})_{\mathbb{K}}$.

In the sequel, we shall call *quasi-representable* any functor $\mathcal{L} : (\mathbf{Wsalg})_{\mathbb{K}} \longrightarrow (\mathbf{Lie})_{\mathbb{K}}$ for which there exists $\mathfrak{g} \in (\mathbf{sLie})_{\mathbb{K}}$ such that $\mathcal{L} \cong \mathcal{L}_{\mathfrak{g}}$; the same applies with $(\mathbf{Grass})_{\mathbb{K}}$ instead of $(\mathbf{Wsalg})_{\mathbb{K}}$.

2.3 Supermanifolds and supergroups

In this subsection we introduce the “supermanifolds” — namely, real smooth, real analytic or complex holomorphic ones — that we work with, as well as the corresponding group objects. The present material is more or less standard; we follow two approaches, that are different but equivalent: we refer to [2] — more or less closely, possibly with different terminology and notation — where all needed details can be found, as well as the original references.

Definition 2.3.1. A *superspace* is a pair $S = (|S|, \mathcal{O}_S)$ of a topological space $|S|$ and a sheaf of commutative superalgebras \mathcal{O}_S on it such that the stalk of \mathcal{O}_S at each point $x \in |S|$, denoted by $\mathcal{O}_{S,x}$, is a local superalgebra. If S and T are two superspaces, a *morphism* $\phi : S \longrightarrow T$ between them is a pair $(|\phi|, \phi^*)$ where $|\phi| : |S| \longrightarrow |T|$ is a continuous map of topological spaces and the induced morphism $\phi^* : \mathcal{O}_T \longrightarrow |\phi|_*(\mathcal{O}_S)$ of sheaves on $|T|$ is such that $\phi^*(\mathfrak{m}_{|\phi|(x)}) \subseteq \mathfrak{m}_x$, where $\mathfrak{m}_{|\phi|(x)}$ and \mathfrak{m}_x denote the maximal ideals in the stalks $\mathcal{O}_{T,|\phi|(x)}$ and $\mathcal{O}_{S,x}$ respectively. \blacklozenge

Examples 2.3.2. Fix any $p, q \in \mathbb{N}_+$.

(a) *The real smooth local model* — The superspace $\mathbb{R}^{p|q}$ is the topological space \mathbb{R}^p endowed with the following sheaf of commutative superalgebras: $\mathcal{O}_{\mathbb{R}^{p|q}}(U) := \mathcal{C}_{\mathbb{R}^p}^{\infty}(U) \otimes_{\mathbb{R}} \Lambda_{\mathbb{R}}(\vartheta_1, \dots, \vartheta_q)$ for

any open set $U \subseteq \mathbb{R}^p$, where $\Lambda_{\mathbb{R}}(\vartheta_1, \dots, \vartheta_q)$ is the real Grassmann algebra on q variables $\vartheta_1, \dots, \vartheta_q$ and $\mathcal{C}_{\mathbb{R}^p}^{\infty}$ is the sheaf of smooth real functions on \mathbb{R}^p .

(b) *The real analytic local model* — The superspace $\mathbb{R}_{\omega}^{p|q}$ is the topological space \mathbb{R}^p endowed with the following sheaf of commutative superalgebras: $\mathcal{O}_{\mathbb{R}_{\omega}^{p|q}}(U) := \mathcal{C}_{\mathbb{R}^p}^{\omega}(U) \otimes_{\mathbb{R}} \Lambda_{\mathbb{R}}(\vartheta_1, \dots, \vartheta_q)$ for any open set $U \subseteq \mathbb{R}^p$, where $\Lambda_{\mathbb{R}}(\vartheta_1, \dots, \vartheta_q)$ is the real Grassmann algebra on q variables $\vartheta_1, \dots, \vartheta_q$ and $\mathcal{C}_{\mathbb{R}^p}^{\omega}$ is the sheaf of analytic real functions on \mathbb{R}^p .

(c) *The holomorphic local model* — The superspace $\mathbb{C}^{p|q}$ is the topological space \mathbb{C}^p endowed with the following sheaf of commutative superalgebras: $\mathcal{O}_{\mathbb{C}^{p|q}}(U) := \mathcal{C}_{\mathbb{C}^p}(U) \otimes_{\mathbb{C}} \Lambda_{\mathbb{C}}(\vartheta_1, \dots, \vartheta_q)$ for any open set $U \subseteq \mathbb{C}^p$, where $\Lambda_{\mathbb{C}}(\vartheta_1, \dots, \vartheta_q)$ is the complex Grassmann algebra on q variables $\vartheta_1, \dots, \vartheta_q$ and $\mathcal{C}_{\mathbb{C}^p}$ is the sheaf of holomorphic (or analytic) complex functions on \mathbb{C}^p .

Patching together these local models one makes up “supermanifolds”, defined as follows:

Definition 2.3.3.

(a) A (real) smooth supermanifold of (super)dimension $p|q$ is a superspace $M = (|M|, \mathcal{O}_M)$ that is locally isomorphic to $\mathbb{R}^{p|q}$, i. e. for every $x \in |M|$ there exists an open sets $V_x \subseteq |M|$ with $x \in V_x$ and $U \subseteq \mathbb{R}^p$ such that $\mathcal{O}_M|_{V_x} \cong \mathcal{O}_{\mathbb{R}^{p|q}}|_U$, and such that $|M|$ is Hausdorff and second-countable. A morphism of smooth supermanifolds is a morphism of the underlying superspaces.

(b) A (real) analytic supermanifold of (super)dimension $p|q$ is defined like in (a) but with $\mathbb{R}_{\omega}^{p|q}$ replacing $\mathbb{R}^{p|q}$ everywhere: in particular, it is locally isomorphic to $\mathbb{R}_{\omega}^{p|q}$. A morphism of analytic supermanifolds is just a morphism of the underlying superspaces.

(c) A (complex) holomorphic supermanifold of (super)dimension $p|q$ is defined like in (a) but with $\mathbb{C}^{p|q}$ replacing $\mathbb{R}^{p|q}$ everywhere: in particular, it is locally isomorphic to $\mathbb{C}^{p|q}$. A morphism of holomorphic supermanifolds is a morphism of the underlying superspaces.

In either case above, the sheaf \mathcal{O}_M is called the *structure sheaf* of M : to simplify notation, I shall write $\mathcal{O}(M)$ instead of $\mathcal{O}_M(|M|)$ to denote the superalgebra of global sections of \mathcal{O}_M .

We denote the category of (real) smooth, (real) analytic, or (complex) holomorphic supermanifolds by **(ssmfd)**, **(asmfd)**, or **(hsmfd)**, respectively. \diamond

In most cases later on the distinction between the smooth, the analytic or the holomorphic case is immaterial: therefore, in order to minimize repetitions, I shall often refer only to “supermanifolds”.

2.3.4. The reduced submanifold of a supermanifold. Let M be a smooth supermanifold and U an open subset in $|M|$. Let $\mathcal{I}_M(U)$ be the (nilpotent) ideal of $\mathcal{O}_M(U)$ generated by the odd part of the latter: then $\mathcal{O}_M/\mathcal{I}_M$ defines a sheaf of purely even superalgebras over $|M|$, locally isomorphic to $\mathcal{C}^{\infty}(\mathbb{R}^p)$. Then $M_{\mathbf{0}} := (|M|, \mathcal{O}_M/\mathcal{I}_M)$ is a classical smooth manifold, usually called the *reduced smooth manifold* associated with M ; moreover, the standard, built-in projection $s \mapsto \tilde{s} := s + \mathcal{I}_M(U)$ (for all $s \in \mathcal{O}_M(U)$) at the sheaf level corresponds to a natural embedding $M_{\mathbf{0}} \longrightarrow M$, so that $M_{\mathbf{0}}$ can be thought of as an embedded sub(super)manifold of M itself.

A similar construction applies when the supermanifold M is analytic, resp. holomorphic, yielding the notion of “*reduced analytic manifold*”, resp. “*reduced holomorphic manifold*”, $M_{\mathbf{0}}$ of M .

A key feature of this construction is that it is natural, i.e. it provides a *functor* from the category of supermanifolds (of either type: smooth, etc.) to the category of manifolds (of the corresponding type), defined on objects by $M \mapsto M_{\mathbf{0}}$ and on morphisms in a natural way — cf. [2] for details.

Finally, by the very definition of supermanifolds (smooth or analytic or holomorphic), one sees at once that also “classical” manifolds (of either type) can be seen as “supermanifolds”, simply observing that their structure sheaf is one of superalgebras that are actually *totally even*, i.e. with

trivial odd part. Conversely, any supermanifold enjoying the latter property is actually a “classical” manifold, nothing more. In other words, classical manifolds identify with those supermanifolds M that actually coincide with their reduced (sub)manifolds M_0 .

We finish this subsection introducing the notion of “Lie supergroup”:

Definition 2.3.5. A group object in the category (\mathbf{ssmfd}) , or (\mathbf{asmfd}) , or (\mathbf{hsmfd}) , is called (real smooth) Lie supergroup, resp. (real) analytic Lie supergroup, resp. (complex) holomorphic Lie supergroup. These objects, together with the obvious morphisms, form a subcategory among supermanifolds, denoted $(\mathbf{Lsgrp})_{\mathbb{R}}^{\infty}$, resp. $(\mathbf{Lsgrp})_{\mathbb{R}}^{\omega}$, resp. $(\mathbf{Lsgrp})_{\mathbb{C}}^{\omega}$. \diamond

2.4 The functorial point of view

In this subsection we introduce the language of “functor(s) of points” for supermanifolds and Lie supergroups, whose basic idea goes back to Weil’s and Grothendieck’s approach to algebraic geometry. Again, for more details the interested reader can refer to [2].

We begin with some notation. For any two categories \mathbf{A} and \mathbf{B} , with $[\mathbf{A}, \mathbf{B}]$ we denote the category of all functors between \mathbf{A} and \mathbf{B} , the morphisms in $[\mathbf{A}, \mathbf{B}]$ being the natural transformations. As usual \mathbf{A}^{op} will denote the *opposite category* to \mathbf{A} , so that $[\mathbf{A}^{\text{op}}, \mathbf{B}]$ is nothing but the category of contravariant functors from \mathbf{A} to \mathbf{B} .

2.4.1. The functor of points of a supermanifold. Our first kind of “functor of points” is the following. Given a (real) smooth supermanifold $M \in (\mathbf{ssmfd})$, its associated *functor of points* $\mathcal{F}_M : (\mathbf{ssmfd})^{\text{op}} \longrightarrow (\mathbf{set})$ is defined on objects by $\mathcal{F}_M(S) := \text{Hom}(S, M)$ and on morphisms by $\mathcal{F}_M(\phi) : \mathcal{F}_M(S) \longrightarrow \mathcal{F}_M(T)$, $f \mapsto (\mathcal{F}_M(\phi))(f) := f \circ \phi$, for all $S, T \in (\mathbf{ssmfd})$ and $\phi \in \text{Hom}(S, T)$. The elements in $\mathcal{F}_M(S)$ are called the S -points of M . A similar definition holds for analytic, resp. holomorphic, supermanifolds, with (\mathbf{asmfd}) , resp. (\mathbf{hsmfd}) , replacing (\mathbf{ssmfd}) wherever it occurs in the previous definition.

Now consider two supermanifolds M and N : in order to write precise formulas we assume them to be both smooth, but the discussion hereafter makes sense the same if M and N are both analytic or both holomorphic. By definition of “functor of points” Yoneda’s lemma yields a bijection

$$\text{Hom}_{(\mathbf{ssmfd})}(M, N) \longleftrightarrow \text{Hom}_{[(\mathbf{ssmfd})^{\text{op}}, (\mathbf{set})]}(\mathcal{F}_M, \mathcal{F}_N)$$

between morphisms $M \longrightarrow N$ and natural transformations $\mathcal{F}_M \longrightarrow \mathcal{F}_N$ (cf. [19], ch. 3, or [9], ch. 6). Thus we have an immersion $\mathcal{Y} : (\mathbf{ssmfd}) \longrightarrow [(\mathbf{ssmfd})^{\text{op}}, (\mathbf{set})]$ that is full and faithful. The objects in $[(\mathbf{ssmfd})^{\text{op}}, (\mathbf{set})]$ that lie in the image of this immersion — i.e., that arise as functors of points of some supermanifold — are exactly those which are (isomorphic to) representable (ones); indeed, not all objects in $[(\mathbf{ssmfd})^{\text{op}}, (\mathbf{set})]$ are representable, but important *representability criteria* exist — e.g., see [2], Theorem 2.13.

Finally, in this functorial approach the Lie supergroups are characterized as follows: any supermanifold M is actually a Lie supergroup if and only if its functor of points \mathcal{F}_M is actually group-valued, i.e. its target category is (\mathbf{group}) rather than (\mathbf{set}) .

2.4.2. The Weil-Berezin functor of \mathcal{A} -points. We introduce now the notion of \mathcal{A} -point of a supermanifold, following the presentation in [2], §3.2. This looks similar to the functor of points considered above, but in the end it is quite different indeed: in particular, in order to “recover” all the information corresponding to a given supermanifold M one is forced to endow each set of \mathcal{A} -points of M with an extra structure, namely that of an “ A_0 -manifold”, that we shall see later.

In the constructions we mainly refer to smooth supermanifolds but, with minimal changes, they adapt to analytic and holomorphic supermanifolds too (cf. [2]). We begin with the key definitions:

Definition 2.4.3. Let M be a supermanifold (either smooth, etc.).

(a) For every point $x \in |M|$ and every Weil superalgebra $A \in (\mathbf{Wsalg})$ we define the set of A -points near x as given by $M_{A,x} := \text{Hom}_{(\mathbf{salg})}(\mathcal{O}_{M,x}, A)$ and the set of (all) A -points as given by $M_A := \bigsqcup_{x \in |M|} M_{A,x}$. If $x_A \in M_{A,x}$ we call $\tilde{x}_A := x$ the base point of x_A .

(b) We denote by $\mathcal{W}_M : (\mathbf{Wsalg}) \rightarrow (\mathbf{set})$ the functor defined by $A \mapsto \mathcal{W}_M(A) := M_A$ on objects and on morphisms by $\rho \mapsto \mathcal{W}_M(\rho) := \rho^{(M)}$ for every $A, B \in (\mathbf{Wsalg})$ and every $\rho \in \text{Hom}_{(\mathbf{salg})}(A, B)$ with $\rho^{(M)} : M_A \rightarrow M_B$, $x_A \mapsto \rho \circ x_A$. \diamond

2.4.4. Weil-Berezin functors and Schvarts embedding. The above mentioned construction of \mathcal{W}_M for a supermanifold M is clearly natural in M : in other words, it gives rise to a functor $\mathcal{B} : (\mathbf{ssmfd}) \rightarrow [(\mathbf{Wsalg}), (\mathbf{set})]$ given on objects by $M \mapsto \mathcal{B}(M) := \mathcal{W}_M$. As it is explained in [2], §3.3, this functor is an embedding which is faithful but definitely not full: thus it is unfit to describe (\mathbf{ssmfd}) , as it does not identify the latter with a full subcategory of $[(\mathbf{Wsalg}), (\mathbf{set})]$. One instead has to understand what is exactly the (non-full) subcategory of $[(\mathbf{Wsalg}), (\mathbf{set})]$ which is the image of \mathcal{B} , and then consequently adapt \mathcal{B} itself to a “nicer” functor. I shall now shortly sketch the construction that is needed to solve this problem, referring to [2], §4, for further details.

One starts by introducing the notion of A_0 -manifolds. Roughly speaking, given a finite dimensional commutative algebra $A_0 \in (\mathbf{alg})_{\mathbb{K}}$, an A_0 -manifold is a (smooth, etc.) manifold M endowed with an L -atlas, i.e. an atlas of local charts each of which is diffeomorphic (or bianalytic, or biholomorphic) with some open subset of a given finite dimensional A_0 -module L in such a way that the differential of every change of charts is an A_0 -module isomorphism (between local copies of L). Given two such A_0 -modules — for possibly different L ’s — and two A_0 -modules M and N modelled on them, an A_0 -smooth (or analytic, or holomorphic) morphism $\phi : M \rightarrow N$ is any morphism from M to N in the standard sense (smooth, analytic, etc.) such that its differential at each point is A_0 -linear. The resulting category is then denoted $(A_0\text{-}\mathbf{smfd})$, resp. $(A_0\text{-}\mathbf{amfd})$, resp. $(A_0\text{-}\mathbf{hmf})$, possibly with a subscript \mathbb{K} .

For the second step, we gather together all possible A_0 -manifolds for all possible finite dimensional $A_0 \in (\mathbf{alg})_{\mathbb{K}}$. Now consider $A', A'' \in (\mathbf{alg})_{\mathbb{K}}$, a morphism $\rho : A' \rightarrow A''$ between them, an A'_0 -module M' and an A''_0 -module M'' . By scalar restriction (through ρ), M'' is also an A'_0 -module, thus we define a morphism from M' to M'' as being a morphism of A'_0 -manifolds. With this notion of “morphism” among them, the various A_0 -manifolds — for different A_0 ’s — altogether form a new category, denoted by $(A_0\text{-}\mathbf{smfd})$, resp. $(A_0\text{-}\mathbf{amfd})$, resp. $(A_0\text{-}\mathbf{hmf})$, in the smooth, resp. analytic, resp. holomorphic case. We can now introduce our next definition:

Definition 2.4.5. We denote $[(\mathbf{Wsalg}), (A_0\text{-}\mathbf{smfd})]$ the subcategory of $[(\mathbf{Wsalg}), (A_0\text{-}\mathbf{smfd})]$ whose objects are those in $[(\mathbf{Wsalg}), (A_0\text{-}\mathbf{smfd})]$ and whose morphisms are all natural transformations $\phi : \mathcal{G} \rightarrow \mathcal{H}$ such that for every $A \in (\mathbf{Wsalg})$ the induced $\phi_A : \mathcal{G}(A) \rightarrow \mathcal{H}(A)$ is A_0 -smooth. Similarly we define the categories $[(\mathbf{Wsalg}), (A_0\text{-}\mathbf{amfd})]$ and $[(\mathbf{Wsalg}), (A_0\text{-}\mathbf{hmf})]$ respectively in the analytic and in the holomorphic case. \diamond

The motivation for introducing the notion of A_0 -manifolds and the categories $(A_0\text{-}\mathbf{smfd})$, $(A_0\text{-}\mathbf{amfd})$ and $(A_0\text{-}\mathbf{hmf})$ lies in the following three results:

- (1) if M is any supermanifold, for each $A \in (\mathbf{Wsalg})$ the set $\mathcal{W}_M(A)$ can be naturally endowed with a canonical structure of A_0 -manifold;
- (2) if M is a supermanifold and $\rho : A \rightarrow B$ a morphism in (\mathbf{Wsalg}) , then the map $\mathcal{W}_M(A) \xrightarrow{\rho^{(M)}} \mathcal{W}_M(B)$ ($x_A \mapsto \rho \circ x_A$) is a morphism in $(A_0\text{-}\mathbf{smfd})$, resp. $(A_0\text{-}\mathbf{amfd})$, resp. $(A_0\text{-}\mathbf{hmf})$.

(3) if $\phi : M \longrightarrow N$ is a morphism of supermanifolds, then for all $A \in (\mathbf{Wsalg})$ the map $\phi_A : \mathcal{W}_M(A) \longrightarrow \mathcal{W}_N(A)$ ($x_A \mapsto x_A \circ \phi^*$) is a morphism in $[[(\mathbf{Wsalg}), (\mathcal{A}_0\text{-smfd})]]$, resp. in $[[(\mathbf{Wsalg}), (\mathcal{A}_0\text{-amfd})]]$, resp. in $[[(\mathbf{Wsalg}), (\mathcal{A}_0\text{-hmf})]]$.

Thanks to the above, we can correctly introduce Weil-Berezin functors and Shvarts embedding:

Definition 2.4.6.

(a) For every smooth supermanifold $M \in (\mathbf{ssmfd})$, we call *Weil-Berezin (local) “functor of A -points”* of M the functor $\mathcal{W}_M : (\mathbf{Wsalg}) \longrightarrow (\mathcal{A}_0\text{-smfd})$ defined as in Definition 2.4.3(b) — which makes sense thanks to the previous remarks. The same terminology applies, *mutatis mutandis*, in the case of any analytic or holomorphic supermanifold.

(b) We call *Shvarts embedding* the functor $\mathcal{S} : (\mathbf{ssmfd}) \longrightarrow [[(\mathbf{Wsalg}), (\mathcal{A}_0\text{-smfd})]]$, in the smooth case, defined on objects by $M \mapsto \mathcal{W}_M$; and similarly in the analytic and the holomorphic case. \diamond

The key point here is that *Shvarts embedding is a full and faithful embedding*, so that for any two supermanifolds, say smooth, M and N one has

$$\text{Hom}_{(\mathbf{ssmfd})}(M, N) \cong \text{Hom}_{[[(\mathbf{Wsalg}), (\mathcal{A}_0\text{-smfd})]]}(\mathcal{S}(M), \mathcal{S}(N)) = \text{Hom}_{[[(\mathbf{Wsalg}), (\mathcal{A}_0\text{-smfd})]]}(\mathcal{W}_M, \mathcal{W}_N)$$

hence in particular $M \cong N$ if and only if $\mathcal{S}(M) \cong \mathcal{S}(N)$, that is $\mathcal{W}_M \cong \mathcal{W}_N$.

Therefore one can correctly study supermanifolds via their Weil-Berezin functors. However, to do that one still has to be able to characterize those objects in $[[(\mathbf{Wsalg}), (\mathcal{A}_0\text{-smfd})]]$ — in the smooth case, and similarly in the other cases — that actually are (isomorphic to) the Weil-Berezin functors of some supermanifolds; in other words, one needs a characterization of the image of Shvarts embedding, which is actually *not* all of its target category, but a proper subcategory of it. This is the “representability problem”, which we do not really care so much for the present work.

What is still relevant to us, is that the Shvarts embedding \mathcal{S} preserves products, hence also group objects. This means, in the end, that the following holds true (cf. [2], §4):

Proposition 2.4.7.

A supermanifold M is a Lie supergroup if and only if $\mathcal{S}(M) := \mathcal{W}_M$ takes values in the subcategory — among \mathcal{A}_0 -manifolds — of group objects (what we might call “Lie \mathcal{A}_0 -groups”).

2.4.8. The Weil-Berezin approach for reduced submanifolds and the classical case. As we saw in §2.3.4, by the very definition of supermanifolds, either smooth or analytic or holomorphic, one sees at once that also “classical” manifolds (of either type) can be seen as “supermanifolds”, simply observing that their structure sheaf is one of superalgebras which are actually *totally even* (i.e. with trivial odd part). Conversely, any supermanifold enjoying this peculiar property is actually a “classical” manifold, nothing more. In other words, these are exactly all those supermanifolds M which actually coincide with their (classical) reduced subsupermanifold M_0 .

From the functorial point of view, it is clear from definitions that a supermanifold M is actually classical if and only if the associated Weil-Berezin functor $\mathcal{S}(M) := \mathcal{W}_M \in [[(\mathbf{Wsalg}), (\mathbf{set})]]$ actually coincides with its restriction to the (full) subcategory $(\mathbf{alg}) \cap (\mathbf{Wsalg})$; in a nutshell, these are those M such that $\mathcal{W}_M(A) = \mathcal{W}_M(A_0)$ for all $A \in (\mathbf{Wsalg})_{\mathbb{K}}$. If instead one deals with a (general) supermanifold M , then the restriction to $(\mathbf{alg}) \cap (\mathbf{Wsalg})$ of its Weil-Berezin functor coincides with the Weil-Berezin functor (in the previous sense, for classical manifolds) of its associated reduced submanifold M_0 : in a nutshell, $\mathcal{S}(M)|_{(\mathbf{alg}) \cap (\mathbf{Wsalg})} = \mathcal{S}(M_0)$.

2.4.9. The functor of Λ -points. As it is explained in [2], §4.3, one can repeat the construction of the Weil-Berezin functor of any supermanifold and of the consequent Shvarts embedding functor

in a somewhat simpler manner, namely replacing systematically the category $(\mathbf{Wsalg})_{\mathbb{K}}$ with its full subcategory $(\mathbf{Grass})_{\mathbb{K}}$. Thus for each smooth supermanifold $M \in (\mathbf{ssmfd})_{\mathbb{R}}$ the restriction to $(\mathbf{Grass})_{\mathbb{R}}$ of its Weil-Berezin functor yields a new functor $\mathcal{W}_M^{(\Lambda)} : (\mathbf{Grass})_{\mathbb{R}} \longrightarrow (\mathcal{A}_0 - \mathbf{smfd})$, which we call “(Weil-Berezin) *functors of Λ -points of M* ”. As M ranges in $(\mathbf{ssmfd})_{\mathbb{R}}$ all these $\mathcal{W}_M^{(\Lambda)}$ ’s give a new functor $\mathcal{S}^{(\Lambda)} : (\mathbf{ssmfd})_{\mathbb{R}} \longrightarrow [(\mathbf{Grass})_{\mathbb{R}}, (\mathcal{A}_0 - \mathbf{smfd})]$ whose main feature is that it is again a full and faithful embedding, which we call again “Shvarts embedding”.

Similarly one does with analytic or holomorphic supermanifolds.

The outcome is that to study supermanifolds it is enough to consider their functors of Λ -points; in particular, two supermanifolds (of either type) are isomorphic if and only if their corresponding functors of points are. Moreover, the new Shvarts embedding $\mathcal{S}^{(\Lambda)}$ again preserves products, so it takes group objects to group objects: therefore, the characterization of Lie supergroups stated in Proposition 2.4.7 still makes sense reading “ $\mathcal{S}^{(\Lambda)}(M) := \mathcal{W}_M^{(\Lambda)}$ ” instead of “ $\mathcal{S}(M) := \mathcal{W}_M$ ”.

A direct consequence of all this is the following. In the rest of the paper, we shall work with Lie supergroups considered as special functors, i.e. we study M using its Weil-Berezin functor of points \mathcal{W}_M ; or even, conversely, we consider special functors $\mathcal{W} : (\mathbf{Wsalg})_{\mathbb{R}} \longrightarrow (\mathcal{A}_0 - \mathbf{smfd})$ — in the smooth case, say — and then prove that there exists some (smooth) Lie supergroup M such that $\mathcal{W}_M = \mathcal{W}$. Now, in force of the above discussion it makes sense to (try to) follow the same strategy using (\mathbf{Grass}) instead then (\mathbf{Wsalg}) and the functor of Λ -points $\mathcal{W}_M^{(\Lambda)}$ instead of \mathcal{W} . The good news are, indeed, that this is actually feasible, in that all our discussion in the sequel will perfectly makes sense and will be equally correct in both approaches. Thus one can choose to work in the larger framework of Weil superalgebras or in the simpler setup of Grassmann algebras, and in both cases our procedure and results will apply and hold true exactly the same.

3 From Lie supergroups to super Harish-Chandra pairs

In this section we present the notion of super Harish-Chandra pairs, showing how it naturally arises from that of Lie supergroup. Indeed, here “naturally” means that one has functorial constructions that, starting from any Lie supergroup, leads to a special “pair”, whose properties are then singled out to set down the very definition of “super Harish-Chandra pairs”. Overall, this provides a remarkable functor from Lie supergroups to super Harish-Chandra pairs.

3.1 The notion of super Harish-Chandra pair

We present now the notion of *super Harish-Chandra pair*, introduced by Koszul in [17] — but this terminology is first found in [8]. When given out of the blue, it might look somewhat artificial, yet we shall see in the next subsection that it naturally arises from the study of lie supergroups.

Definition 3.1.1. We call *super Harish-Chandra pair* (smooth, analytic or holomorphic) over \mathbb{K} any pair (G_+, \mathfrak{g}) such that

- (a) G_+ is a (smooth, analytic or holomorphic) Lie group over \mathbb{K} , and $\mathfrak{g} \in (\mathbf{sLie})_{\mathbb{K}}$;
- (b) $\mathrm{Lie}(G_+) = \mathfrak{g}_0$;
- (c) there is a (smooth, analytic or holomorphic) G_+ -action on \mathfrak{g} by Lie \mathbb{K} -superalgebra automorphisms, hereafter denoted by $\mathrm{Ad} : G_+ \longrightarrow \mathrm{Aut}(\mathfrak{g})$, such that its restriction to \mathfrak{g}_0 is the adjoint action of G_+ on $\mathrm{Lie}(G_+) = \mathfrak{g}_0$ and the differential of this action is the restriction to $\mathrm{Lie}(G_+) \times \mathfrak{g} = \mathfrak{g}_0 \times \mathfrak{g}$ of the adjoint action of \mathfrak{g} .

If (G'_+, \mathfrak{g}') and (G''_+, \mathfrak{g}'') are two super Harish-Chandra pairs over \mathbb{K} , a *morphism* among them is any pair $(\Omega_+, \omega) : (G'_+, \mathfrak{g}') \longrightarrow (G''_+, \mathfrak{g}'')$ where $\Omega_+ : G'_+ \longrightarrow G''_+$ is a morphism of Lie groups (in the smooth, analytic or holomorphic sense), $\omega : \mathfrak{g}' \longrightarrow \mathfrak{g}''$ is a morphism of Lie superalgebras, and the two are compatible with the additional structure, that is to say

$$(d) \quad \omega|_{\mathfrak{g}_0} = d\Omega_+ \quad , \quad Ad(\Omega_+(g)) \circ \omega = \omega \circ Ad(g) \quad \forall \, g \in G_+ \quad .$$

All super Harish-Chandra pairs over \mathbb{K} form the objects of a category, denoted $(\mathbf{sHCp})_{\mathbb{K}}$. When we have to specify its type, we write $(\mathbf{sHCp})_{\mathbb{R}}^{\infty}$ if this type is real smooth, $(\mathbf{sHCp})_{\mathbb{R}}^{\omega}$ if it is real analytic, and $(\mathbf{sHCp})_{\mathbb{C}}^{\omega}$ if it is complex holomorphic. \diamond

3.2 Super Harish-Chandra pairs from Lie supergroups

In this subsection we show how one can naturally associate a significant super Harish-Chandra pair with any Lie supergroup; indeed, this is the natural reason why the very notion of super Harish-Chandra pair was introduced. Whatever follows is well-known, for details and proofs the reader can refer to [6] or other similar sources.

3.2.1. The reduced subgroup of a Lie supergroup. Let G be a Lie supergroup (of either type: smooth, etc.). As it is a supermanifold, from §2.3.4 we know that there exists also a reduced submanifold G_0 of G . Taking the functorial point of view, we know that $\mathcal{S}_{G_0} = \mathcal{S}_G|_{(\mathbf{alg})} \cap (\mathbf{Wsalg})$ (cf. §2.4.8) and \mathcal{S}_G takes values in the subcategory of Lie \mathcal{A}_0 -groups, (by Proposition 2.4.7); but then the latter is true for \mathcal{S}_{G_0} as well, hence — by Proposition 2.4.7 again — we argue that G_0 *itself is indeed a Lie group* (either smooth, etc., like G is).

Moreover, when $\phi : G' \longrightarrow G''$ is a morphism of Lie supergroups, the morphism of manifolds $\phi_0 : G'_0 \longrightarrow G''_0$ — induced by the functoriality of the construction $G \mapsto G_0$ — is in addition a Lie group morphism. Therefore, we conclude that $G \mapsto G_0$ and $\phi \mapsto \phi_0$ define a functor from Lie supergroups (of either type) to Lie groups (of the same type).

3.2.2. The tangent Lie superalgebra of a Lie supergroup. We now quickly recall how to associate with a Lie supergroup its “tangent Lie superalgebra”, referring to [6] for further details.

For any $A \in (\mathbf{Wsalg})_{\mathbb{K}}$, we let $A[\varepsilon] := A[x]/(x^2)$ be the so-called *superalgebra of dual numbers* over A , in which $\varepsilon := x \bmod (x^2)$ is taken to be *even*. Then $A[\varepsilon] = A \oplus A\varepsilon$, and there are two natural morphisms $i_A : A \longrightarrow A[\varepsilon]$, $a \mapsto a$, and $p_A : A[\varepsilon] \longrightarrow A$, $(a + a'\varepsilon) \mapsto a$, such that $p_A \circ i_A = \text{id}_A$. Note that it follows by construction that $A[\varepsilon] \in (\mathbf{Wsalg})_{\mathbb{K}}$ again.

Definition 3.2.3. Given a functor $G : (\mathbf{Wsalg})_{\mathbb{K}} \longrightarrow (\mathbf{group})$, let $G(p_A) : G(A[\varepsilon]) \longrightarrow G(A)$ be the morphism associated with the morphism $p_A : A[\varepsilon] \longrightarrow A$ in $(\mathbf{Wsalg})_{\mathbb{K}}$. Then there exists a unique functor $\text{Lie}(G) : (\mathbf{Wsalg})_{\mathbb{K}} \longrightarrow (\mathbf{set})$ given on objects by $\text{Lie}(G)(A) := \text{Ker}(G(p)_A)$. \diamond

The key fact is that when the functor G as above is a (smooth, analytic or holomorphic) Lie supergroup, then $\text{Lie}(G)$ is a Lie algebra valued, i.e. it is a functor $\text{Lie}(G) : (\mathbf{Wsalg})_{\mathbb{K}} \longrightarrow (\mathbf{Lie})_{\mathbb{K}}$. This requires a non-trivial proof (like in the classical case), for which we refer to [6], Ch. 11 (with the few adaptations needed for the present setup), and only quickly sketch here the main steps.

The Lie structure on any object $\text{Lie}(G)(A)$ is introduced as follows. First, define the *adjoint action* of G on $\text{Lie}(G)$ as given, for every $A \in (\mathbf{Wsalg})_{\mathbb{K}}$, by

$$Ad : G(A) \longrightarrow \text{GL}(\text{Lie}(G)(A)) \quad , \quad Ad(g)(x) := G(i)(g) \cdot x \cdot (G(i)(g))^{-1}$$

for all $g \in G(A)$, $x \in \text{Lie}(G)(A)$. Second, define the *adjoint morphism* ad as

$$\text{ad} := \text{Lie}(\text{Ad}) : \text{Lie}(G) \longrightarrow \text{Lie}(\text{GL}(\text{Lie}(G))) := \text{End}(\text{Lie}(G))$$

and finally define $[x, y] := \text{ad}(x)(y)$ for all $x, y \in \text{Lie}(G)(A)$. Then we have the following:

Proposition 3.2.4. *Given a (smooth, analytic or holomorphic) Lie supergroup G , let $\mathfrak{g} := T_e(G)$ be the tangent \mathbb{K} -supermodule to G at the unit point $e \in G$.*

- (a) *$\text{Lie}(G)$ with the bracket $[\cdot, \cdot]$ above is Lie algebra valued, i.e. $\text{Lie}(G) : (\mathbf{Wsalg})_{\mathbb{K}} \longrightarrow (\mathbf{Lie})_{\mathbb{K}}$;*
- (b) *$\text{Lie}(G)$ is quasi-representable (see §2.2.3), namely $\text{Lie}(G) = \mathcal{L}_{\mathfrak{g}}$, where \mathfrak{g} is endowed with a canonical structure of Lie \mathbb{K} -superalgebra, and it is also representable, namely represented by \mathfrak{g}^* ;*
- (c) *for every $A \in (\mathbf{Wsalg})_{\mathbb{K}}$ one has $\text{Lie}(G)(A) = \text{Lie}(G(A))$, the latter being the tangent Lie algebra to the Lie group $G(A)$.*

Note that underneath the previous proposition there is all that was explained in §2.2.3 about Lie superalgebras and their functorial presentation/characterization.

N.B.: in force of the previous proposition, in the sequel we shall freely identify the functor $\text{Lie}(G) = \mathcal{L}_{\mathfrak{g}}$ with the tangent superspace \mathfrak{g} — now thought of as a Lie superalgebra — calling this common object “the *tangent Lie superalgebra* of (or “to”) the Lie supergroup G ”.

There exist several other realizations of the tangent Lie superalgebra to G , and canonical identifications among all of them. We shall only occasionally need some of them, so we do not go into further details, but refer instead to the literature, in particular [7] (especially §5.2 therein).

Finally, the construction $G \mapsto \text{Lie}(G)$ for Lie supergroups is actually natural, in that any morphism $\phi : G' \longrightarrow G''$ of Lie supergroups induces a morphism $\text{Lie}(\phi) : \text{Lie}(G') \longrightarrow \text{Lie}(G'')$ of Lie superalgebras. Eventually, all this together provides functors $\text{Lie} : (\mathbf{Lsgrp})_{\mathbb{R}}^{\infty} \longrightarrow (\mathbf{sLie})_{\mathbb{R}}$, $\text{Lie} : (\mathbf{Lsgrp})_{\mathbb{R}}^{\omega} \longrightarrow (\mathbf{sLie})_{\mathbb{R}}$ and $\text{Lie} : (\mathbf{Lsgrp})_{\mathbb{C}}^{\omega} \longrightarrow (\mathbf{sLie})_{\mathbb{C}}$; see [6] and [2] for details.

3.2.5. The super Harish-Chandra pair of a Lie supergroup. Gathering together the previous results about the reduced subgroup and the Lie superalgebra associated with a Lie supergroup we end up with the core of the present section. Namely, if G is any Lie supergroup then $(G_0, \text{Lie}(G))$ is a super Harish-Chandra pair, and this construction is functorial, as the following claims:

Theorem 3.2.6. *(see for instance [7]) There exist functors*

$$\Phi : (\mathbf{Lsgrp})_{\mathbb{R}}^{\infty} \longrightarrow (\mathbf{sHCp})_{\mathbb{R}}^{\infty}, \quad \Phi : (\mathbf{Lsgrp})_{\mathbb{R}}^{\omega} \longrightarrow (\mathbf{sHCp})_{\mathbb{R}}^{\omega}, \quad \Phi : (\mathbf{Lsgrp})_{\mathbb{C}}^{\omega} \longrightarrow (\mathbf{sHCp})_{\mathbb{C}}^{\omega}$$

that are given on objects by $G \mapsto (G_0, \text{Lie}(G))$ and on morphisms by $\phi \mapsto (\phi_0, \text{Lie}(\phi))$.

4 Interlude: special splittings for Lie supergroups

In this section we present some results concerning the possibility to split Lie supergroups in some (more or less canonical) “special” ways: Boseck’s splittings and global splittings. These results are essentially well-known, yet possibly formulated in different ways; however, we provide independent proofs for them, so to fill any possible gap in literature and mostly to have a self-contained presentation, dealing with Lie supergroups of all types (smooth, analytic and holomorphic).

4.1 Boseck's splitting for Lie supergroups

This subsection is devoted to a first kind of splitting that we refer to as “Boseck's splitting”, as it was first mentioned in Boseck's work [5]. The starting point is the following easy result:

Lemma 4.1.1. *Let $p : A' \longrightarrow A''$ and $u : A'' \longrightarrow A'$ be morphisms in $(\mathbf{Wsalg})_{\mathbb{K}}$ such that $p \circ u = \text{id}_{A''}$ (hence p_A is surjective and u_A injective), and let $G : (\mathbf{Wsalg})_{\mathbb{K}} \longrightarrow (\mathbf{group})$ be any functor. Then $G(A)$ canonically splits into a semi-direct product*

$$G(A) = \text{Im}(G(u)) \ltimes \text{Ker}(G(p)) \cong G(A'') \ltimes \text{Ker}(G(p))$$

Proof. From $p \circ u = \text{id}_{A''}$ we get $G(p) \circ G(u) = G(p \circ u) = G(\text{id}_{A''}) = \text{id}_{G(A'')}$; thus $G(u)$, resp. $G(p)$, is a section of $G(p)$, resp. a retraction of $G(u)$, in the category of groups: in particular, $G(u)$ is injective and $G(p)$ surjective. The claim then follows by standard group-theoretic arguments. \square

When the group-valued functor G is in fact a Lie supergroup, the previous result yields an interesting outcome, as follows:

Proposition 4.1.2. (cf. [5], §2, Proposition 7)

Let G be any (smooth, analytic or holomorphic) Lie supergroup over \mathbb{K} . Then for every Weil superalgebra $A \in (\mathbf{Wsalg})_{\mathbb{K}}$ there exists a canonical splitting of Lie groups

$$G(A) \cong G_0(\mathbb{K}) \ltimes N_G(A) \quad (4.1)$$

where $G(\mathbb{K}) = G_0(\mathbb{K})$ is nothing but the classical, ordinary Lie group underlying G — i.e., the Lie group of \mathbb{K} -points of G_0 — and $N_G(A) := \text{Ker}(G(p_A))$ where $p_A : A \longrightarrow \mathbb{K}$ is the built-in projection of A onto \mathbb{K} (cf. Definition 2.1.3).

Proof. By assumption, for the given $A \in (\mathbf{Wsalg})_{\mathbb{K}}$ we have morphisms $p_A : A \longrightarrow \mathbb{K}$ and $u_A : \mathbb{K} \longrightarrow A$ in $(\mathbf{Wsalg})_{\mathbb{K}}$ such that $p_A \circ u_A = \text{id}_{\mathbb{K}}$ (cf. Definition 2.1.3). Then Lemma 4.1.1 applies, thus yielding a group-theoretic splitting $G(A) = \text{Im}(G(u_A)) \ltimes \text{Ker}(G(p_A))$. Now, as G takes values into the category of Lie groups, this is actually a splitting inside that category — i.e., this a splitting of Lie groups (actually, even one of \mathcal{A}_0 -Lie groups indeed). Moreover, we clearly have $\text{Im}(G(u_A)) \cong G(\mathbb{K}) = G_0(\mathbb{K})$, whence (4.1) is proved. \square

It is worth stressing that this result also has a specific application for the case of *totally even* supergroups — in other words, classical Lie groups — as follows:

Proposition 4.1.3. *Let G_+ be any (smooth, analytic or holomorphic) Lie group over \mathbb{K} . Then for every $A_+ \in (\mathbf{Wsalg})_{\mathbb{K}} \cap (\mathbf{alg})_{\mathbb{K}}$ there exists a canonical splitting of Lie groups*

$$G_+(A_+) \cong G_+(\mathbb{K}) \ltimes N_{G_+}(A_+) \quad (4.2)$$

where $G_+(\mathbb{K})$ is the ordinary Lie group underlying G (i.e., the Lie group of \mathbb{K} -points of G_+) and $N_{G_+}(A_+) := \text{Ker}(G(p_{A_+}))$ with $p_{A_+} : A_+ \longrightarrow \mathbb{K}$ as in Definition 2.1.3.

Proof. The same arguments as for the proof of Proposition 4.1.2 apply again. \square

Remark 4.1.4. To the best of the author's knowledge, the (canonical) splitting (4.1) of $G(A)$ was first mentioned by Boseck (dealing with Lie supergroups defined over $(\mathbf{Grass})_{\mathbb{K}}$, but the idea is the same): cf. [5], §2, Proposition 7; thus we shall refer to (4.1) or (4.2) as to “Boseck's splitting(s)”. The same result was considered by other authors too, e.g. Molotkov: see (7.4.1) in §7.4 of [22].

4.1.5. Bosek's splitting for Lie superalgebras. The notion of “Bosek's splitting” for Lie supergroups has a natural counterpart for Lie superalgebras, when thought of as functors.

Indeed, consider a Lie \mathbb{K} -superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ and the functor $\mathcal{L}_{\mathfrak{g}} : (\mathbf{salg})_{\mathbb{K}} \longrightarrow (\mathbf{Lie})_{\mathbb{K}}$ given by $\mathcal{L}_{\mathfrak{g}}(A) := (A \otimes \mathfrak{g})_0 = (A_0 \otimes \mathfrak{g}_0) \oplus (A_1 \otimes \mathfrak{g}_1)$, for all $A \in (\mathbf{salg})_{\mathbb{K}}$, as in §2.2.3. Every $A \in (\mathbf{Wsalg})_{\mathbb{K}}$ has built-in morphisms $p_A : A \longrightarrow \mathbb{K}$ and $u_A : \mathbb{K} \longrightarrow A$ such that $p_A \circ u_A = \text{id}_{\mathbb{K}}$. Then applying $\mathcal{L}_{\mathfrak{g}}$ we get $\mathcal{L}_{\mathfrak{g}}(p_A) \circ \mathcal{L}_{\mathfrak{g}}(u_A) = \mathcal{L}_{\mathfrak{g}}(p_A \circ u_A) = \mathcal{L}_{\mathfrak{g}}(\text{id}_{\mathbb{K}}) = \text{id}_{\mathcal{L}_{\mathfrak{g}}(\mathbb{K})}$, a relation regarding morphisms of Lie algebras. By standard arguments this yields a Lie algebra splitting

$$\mathcal{L}_{\mathfrak{g}}(A) = \text{Im}(\mathcal{L}_{\mathfrak{g}}(u_A)) \oplus \text{Ker}(\mathcal{L}_{\mathfrak{g}}(p_A)) \quad (4.3)$$

where the symbol “ \oplus ” denotes the (internal) semi-direct sum of $\text{Im}(\mathcal{L}_{\mathfrak{g}}(u_A))$ — a Lie subalgebra inside $\mathcal{L}_{\mathfrak{g}}(A)$ — with $\text{Ker}(\mathcal{L}_{\mathfrak{g}}(p_A))$ — a Lie ideal in $\mathcal{L}_{\mathfrak{g}}(A)$. Now, on the one hand definitions give $\text{Im}(\mathcal{L}_{\mathfrak{g}}(u_A)) \cong \mathcal{L}_{\mathfrak{g}}(\mathbb{K}) := (\mathbb{K} \otimes_{\mathbb{K}} \mathfrak{g})_0 = \mathfrak{g}_0$; on the other hand, to simplify a bit the notation we write $\mathfrak{n}_{\mathfrak{g}}(A) := \text{Ker}(\mathcal{L}_{\mathfrak{g}}(p_A))$. Then (4.3) reads also

$$\mathcal{L}_{\mathfrak{g}}(A) = \mathfrak{g}_0 \oplus \mathfrak{n}_{\mathfrak{g}}(A) \quad \forall A \in (\mathbf{Wsalg})_{\mathbb{K}} \quad (4.4)$$

In the following, we shall refer to (4.4) as to “Bosek's splitting for $\mathcal{L}_{\mathfrak{g}}$ ” — or simply “for \mathfrak{g} ” itself.

It is still worth remarking that one has a non-trivial Bosek's splitting also when $\mathfrak{g} = \mathfrak{g}_0$, i.e. \mathfrak{g} is a classical (=totally even) Lie algebra. Indeed, if \mathfrak{g}_+ is just a Lie \mathbb{K} -algebra then (4.4) reads

$$\mathcal{L}_{\mathfrak{g}_+}(A_+) = \mathfrak{g}_+ \oplus \mathfrak{n}_{\mathfrak{g}_+}(A_+) \quad \forall A_+ \in (\mathbf{Wsalg})_{\mathbb{K}} \cap (\mathbf{alg})_{\mathbb{K}} \quad (4.5)$$

and will again be called “Bosek's splitting for $\mathcal{L}_{\mathfrak{g}_+}$ ” — or “for \mathfrak{g}_+ ”.

We shall now give an explicit description of $\mathfrak{n}_{\mathfrak{g}}(A)$. By definition, $\mathfrak{n}_{\mathfrak{g}}(A) := \text{Ker}(\mathcal{L}_{\mathfrak{g}}(p_A))$ where $p_A : A = \mathbb{K} \oplus \overset{\circ}{A} \longrightarrow \mathbb{K}$ is the canonical projection of $A = \mathbb{K} \oplus \overset{\circ}{A}$ onto its left-hand side summand. Now, $A = A_0 \oplus A_1$ with $A_0 = \mathbb{K} \oplus \overset{\circ}{A}_0$ and $A_1 = \overset{\circ}{A}_1$, hence

$$\mathcal{L}_{\mathfrak{g}}(A) := (A \otimes_{\mathbb{K}} \mathfrak{g})_0 = (A_0 \otimes_{\mathbb{K}} \mathfrak{g}_0) \oplus (A_1 \otimes_{\mathbb{K}} \mathfrak{g}_1) = \mathfrak{g}_0 \oplus (\overset{\circ}{A}_0 \otimes_{\mathbb{K}} \mathfrak{g}_0) \oplus (A_1 \otimes_{\mathbb{K}} \mathfrak{g}_1)$$

from which it clearly follows that

$$\mathfrak{n}_{\mathfrak{g}}(A) := \text{Ker}(\mathcal{L}_{\mathfrak{g}}(p_A)) = (\overset{\circ}{A}_0 \otimes_{\mathbb{K}} \mathfrak{g}_0) \oplus (\overset{\circ}{A}_1 \otimes_{\mathbb{K}} \mathfrak{g}_1) \quad \forall A \in (\mathbf{Wsalg})_{\mathbb{K}} \quad (4.6)$$

In the case when $\mathfrak{g} = \mathfrak{g}_0$ is just a “classical” Lie algebra, say \mathfrak{g}_+ , this reads slightly simpler, namely

$$\mathfrak{n}_{\mathfrak{g}_+}(A_+) = \overset{\circ}{A}_+ \otimes_{\mathbb{K}} \mathfrak{g}_+ \quad \forall A_+ \in (\mathbf{Wsalg})_{\mathbb{K}} \cap (\mathbf{alg})_{\mathbb{K}} \quad (4.7)$$

This entails the following:

Proposition 4.1.6.

- (a) Let \mathfrak{g} be a Lie \mathbb{K} -superalgebra and $A \in (\mathbf{Wsalg})_{\mathbb{K}}$. Then the Lie algebra $\mathfrak{n}_{\mathfrak{g}}(A)$ is nilpotent.
- (b) Let \mathfrak{g}_+ be a Lie \mathbb{K} -algebra and let $A_+ \in (\mathbf{Wsalg})_{\mathbb{K}} \cap (\mathbf{alg})_{\mathbb{K}}$. Then $\mathfrak{n}_{\mathfrak{g}_+}(A_+)$ is nilpotent.

Proof. Claim (a) follows at once from (4.6) and the fact that $\overset{\circ}{A}$ is nilpotent, and likewise for (b). \square

4.1.7. The interplay of Bosek's splittings for supergroups and superalgebras. Let again G be a Lie supergroup over \mathbb{K} , and $\mathfrak{g} := \text{Lie}(G)$ be its tangent Lie superalgebra. For any $A \in (\mathbf{Wsalg})_{\mathbb{K}}$, the Lie group $G(A)$ and the Lie algebra $\mathfrak{g}(A) := \mathcal{L}_{\mathfrak{g}}(A)$ — also equal to $\text{Lie}(G)(A) = \text{Lie}(G(A))$, cf. Proposition 3.2.4(c) — are linked by the exponential map

$$\exp : \mathfrak{g}(A) \longrightarrow G(A)$$

which is a local isomorphism (either in the smooth, analytic or holomorphic sense, as usual). Similarly for the “even counterparts” we have the local isomorphism as well

$$\exp_0 : \mathfrak{g}_0 \longrightarrow G_0(\mathbb{K})$$

with $\exp_0 = \exp|_{\mathfrak{g}_0}$ if we think at \mathfrak{g}_0 as embedded into $\mathfrak{g}(A) := \mathcal{L}_{\mathfrak{g}}(A) = \mathfrak{g}_0 \oplus \mathfrak{n}_{\mathfrak{g}}(A)$ — cf. (4.4).

Now, since the Lie algebra $\mathfrak{n}_{\mathfrak{g}}(A)$ is nilpotent — cf. Proposition 4.1.6 — its image $\exp(\mathfrak{n}_{\mathfrak{g}}(A))$ for the exponential map is a (closed, connected) nilpotent Lie subgroup of $G(A)$. Furthermore, let us use notation $\mathfrak{g}(p_A) := \mathcal{L}_{\mathfrak{g}}(p_A)$ and $\mathfrak{g}(u_A) := \mathcal{L}_{\mathfrak{g}}(u_A)$, and consider the diagram

$$\begin{array}{ccc} \mathfrak{g}(A) & \xrightarrow{\exp} & G(A) \\ \mathfrak{g}(u_A) \updownarrow \mathfrak{g}(p_A) & & G(p_A) \updownarrow G(u_A) \\ \mathfrak{g}_0 & \xrightarrow{\exp_0} & G_0(\mathbb{K}) \end{array} \quad (4.8)$$

This diagram is commutative, hence in particular $G(p_A) \circ \exp = \exp_0 \circ \mathfrak{g}(p_A)$, which in turn implies at once $G(p_A)(\exp(\mathfrak{n}_{\mathfrak{g}}(A))) = \exp_0(\mathfrak{g}(p_A)(\mathfrak{n}_{\mathfrak{g}}(A))) = \exp_0(\{0_{\mathfrak{g}_0}\}) = \{1_{G_0(\mathbb{K})}\}$ because $\mathfrak{n}_{\mathfrak{g}}(A) := \text{Ker}(\mathfrak{g}(p_A))$; so in the end $\exp(\mathfrak{n}_{\mathfrak{g}}(A)) \subseteq \text{Ker}(G(p_A)) =: N_G(A)$ — cf. Proposition 4.1.

The fact that $\exp : \mathfrak{g}(A) \longrightarrow G(A)$ be a local isomorphism, together with Boseck’s splittings — namely, $\mathfrak{g}(A) = \mathfrak{g}_0 \oplus \mathfrak{n}_{\mathfrak{g}}(A)$ and $G(A) = G_0(\mathbb{K}) \ltimes N_G(A)$ — and $\dim(\mathfrak{g}_0) = \dim(G_0(\mathbb{K}))$, jointly imply that $\dim(\mathfrak{n}_{\mathfrak{g}}(A)) = \dim(N_G(A))$. On the other hand, since $\mathfrak{n}_{\mathfrak{g}}(A)$ is nilpotent we know that its exponential map — that is just the restriction to $\mathfrak{n}_{\mathfrak{g}}(A)$ of $\exp : \mathfrak{g}(A) \longrightarrow G(A)$ — is actually a *global isomorphism* of \mathbb{K} -manifolds from $\mathfrak{n}_{\mathfrak{g}}(A)$ to $\exp(\mathfrak{n}_{\mathfrak{g}}(A))$. As a byproduct then we have $\dim(\exp(\mathfrak{n}_{\mathfrak{g}}(A))) = \dim(N_G(A))$.

Let now $N_G(A)^\circ$ denote the connected component of $N_G(A)$, for which $\dim(N_G(A)^\circ) = \dim(N_G(A))$; our previous analysis, here above, guarantees that $\exp(\mathfrak{n}_{\mathfrak{g}}(A)) \subseteq N_G(A)^\circ$, the former being a closed Lie subgroup inside the latter. But $\dim(\exp(\mathfrak{n}_{\mathfrak{g}}(A))) = \dim(N_G(A)^\circ)$ too, hence eventually we conclude that $\exp(\mathfrak{n}_{\mathfrak{g}}(A)) = N_G(A)^\circ$.

As our next step we shall provide an thorough analysis of the Lie group $\exp(\mathfrak{n}_{\mathfrak{g}}(A)) = N_G(A)^\circ$, in particular finding that it actually coincides with $N_G(A)$.

4.1.8. The Lie subgroup $N_G(A)$. Let G be a Lie supergroup, as before. As we saw in §2.4, for any $A \in (\mathbf{Wsalg})_{\mathbb{K}}$ the group $G(A)$ is defined as

$$G(A) := G_A = \bigsqcup_{g \in |G|} G_{A,g}$$

where $|G|$ is the underlying topological space of G and $G_{A,g} := \text{Hom}_{(\mathbf{salg})_{\mathbb{K}}}(\mathcal{O}_{G,g}, A)$, with $\mathcal{O}_{G,g}$ being the stalk (a local superalgebra) of the structure sheaf of G at the point $g \in |G|$. Moreover, we adopt the canonical identification $|G| = G(\mathbb{K})$ via $g \mapsto \text{ev}_g$ with $\text{ev}_g : \mathcal{O}_{G,g} \longrightarrow \mathbb{K}$ given by $f \mapsto \text{ev}_g(f) := f(g)$. As additional notation, for every $g_A \in \mathcal{O}_{G,g}$ we set $\tilde{g}_A := g$ ($= \text{ev}_g$).

In force of the splitting $A = \mathbb{K} \oplus \mathring{A}$, for every $g_A \in G(A)$, say $g_A \in G_{A,g}$, there exists a unique map $\hat{g}_A : \mathcal{O}_{G,g} \longrightarrow \mathring{A}$ such that $g_A = \tilde{g}_A + \hat{g}_A$. Then, by construction, for all $f \in \mathcal{O}_{G,g}$ we have

$$\tilde{g}_A(f) = p_A(\tilde{g}_A(f) + \hat{g}_A(f)) = p_A(g(f)) = (p_A \circ g)(f) = (G(p_A)(g))(f)$$

hence we get $\tilde{g}_A = G(p_A)(g)$.

Now assume $g_A \in N_G(A) := \text{Ker}(G(p_A))$. Then $G(p_A)(g_A) = 1_{G_A} \in G(A)$; therefore — by the previous analysis — we have $g_A = 1 + \hat{g}_A$ (which can be read as the sum, in the natural sense, of maps from $\mathcal{O}_{G,1}$ to A). We can re-write our g_A as

$$g_A = 1 + \hat{g}_A = \exp(X_{g_A}) \quad \text{with} \quad X_{g_A} := \log(g_A) = \sum_{n=1}^{+\infty} (-1)^{n+1} \frac{\hat{g}_A^n}{n} \quad (4.9)$$

where $\exp(X_{g_A}) := \sum_{n=0}^{+\infty} X_{g_A}^n / n!$ and all powers (and products) involved in these formulas are simply given by $X_{g_A}^n(f) := (X_{g_A}(f))^n$, $\hat{g}_A^n(f) := (\hat{g}_A(f))^n$, etc. Note that all this does make sense because $\text{Im}(\hat{g}_A) \in \mathring{A}$, by construction; thus \hat{g}_A itself is nilpotent, hence X_{g_A} is given by a finite sum and it is in turn nilpotent, so the formal series defining $\exp(X_{g_A})$ is a finite sum as well.

By formal properties of exponential and logarithm, since $g_A : \mathcal{O}_{G,1} \rightarrow A$ is a (superalgebra) morphism it follows from (4.9) that $X_{g_A} : \mathcal{O}_{G,1} \rightarrow A$ is in turn a (superalgebra) derivation; thus — cf. §Proposition 3.2.4(c) — we have $X_{g_A} \in \text{Lie}(G(A)) = (\text{Lie}(G))(A) = \mathcal{L}_{\mathfrak{g}}(A) =: \mathfrak{g}(A)$. Finally, by construction we have also $\text{Im}(X_{g_A}) \in \mathring{A}$. Along with Boseck's splitting $\mathfrak{g}(A) = \mathfrak{g}_0 \oplus \mathfrak{g}_1(A)$ — see (4.4) for $\mathfrak{g}(A) := \mathcal{L}_{\mathfrak{g}}(A)$ — and with $\mathfrak{n}_{\mathfrak{g}}(A) = (\mathring{A}_0 \otimes_{\mathbb{K}} \mathfrak{g}_0) \oplus (\mathring{A}_1 \otimes_{\mathbb{K}} \mathfrak{g}_1)$ — as in (4.6) — all this together eventually leads to $X_{g_A} \in \mathfrak{n}_{\mathfrak{g}}(A)$. Tiding everything up, we come now to the end:

Proposition 4.1.9. *For every Lie supergroup G and every $A \in (\mathbf{Wsalg})_{\mathbb{K}}$ we have*

$$N_G(A) = \exp(\mathfrak{n}_{\mathfrak{g}}(A))$$

In particular, $N_G(A)$ is connected nilpotent, and (globally) isomorphic, as a manifold, to $\mathfrak{n}_{\mathfrak{g}}(A)$.

Proof. Indeed, our analysis above shows that each $g_A \in N_G(A)$ can be realized as $g_A = \exp(X_{g_A})$ with $X_{g_A} \in \mathfrak{n}_{\mathfrak{g}}(A)$; hence $N_G(A) \subseteq \exp(\mathfrak{n}_{\mathfrak{g}}(A))$. On the other hand, from §4.1.7 we have also $\exp(\mathfrak{n}_{\mathfrak{g}}(A)) = N_G(A)^{\circ} \subseteq N_G(A)$. Thus $N_G(A) = \exp(\mathfrak{n}_{\mathfrak{g}}(A))$ as claimed.

The last part of the claim then is clear. \square

4.2 Global splittings for Lie supergroups

This subsection is devoted to find yet another remarkable splitting for the groups $G(A)$ of A -points of any Lie supergroup G ; this is no longer canonical — as it depends on some choice to be made — but on the other hand it is natural in A , hence overall means that one has a noteworthy splitting for G itself as a functor, usually known as “global splitting” of G . However, such a result is often stated in a form which is not as “geometric” as we wished — typically, as a splitting of the structure sheaf (cf. for instance: [3], Ch. 2 §2; [22], §7.4; [24], §2) — so we prefer to recover such a result once more, stating it in an explicitly geometrical form.

The inspiring idea underneath the whole construction is that we look for a splitting of the form $G(A) = G_0(A) \times G^{\leq}(A)$ which has to be, somehow, a group-theoretic counterpart of the splitting of $\mathfrak{g}(A) := \mathcal{L}_{\mathfrak{g}}(A)$ into $\mathfrak{g}(A) = (\mathring{A}_0 \otimes_{\mathbb{K}} \mathfrak{g}_0) \oplus (\mathring{A}_1 \otimes_{\mathbb{K}} \mathfrak{g}_1)$. Indeed, we shall achieve such a goal relying upon Boseck's splitting considered in §4.1 above.

4.2.1. Structure theorem and global splitting(s) for Lie supergroups. As above let G be a (smooth, analytic or holomorphic) Lie supergroup over \mathbb{K} , whose tangent Lie superalgebra is $\mathfrak{g} := \text{Lie}(G)$, and let $A \in (\mathbf{Wsalg})_{\mathbb{K}}$ any Weil superalgebra. If we consider the powers \mathring{A}^d of the

nilradical \mathring{A} of A , they form a descending sequence that, by assumption (see Definition 2.1.3), one has $\mathring{A}^N = 0$ for $N \gg 0$. Associated with this, we can consider the sequence

$$\mathfrak{n}_{\mathfrak{g}}^{(d)}(A) := \left(\mathring{A}^d \otimes_{\mathbb{K}} \mathfrak{g} \right)_{\mathbf{0}} = \left((\mathring{A}^d)_{\mathbf{0}} \otimes_{\mathbb{K}} \mathfrak{g} \right) \oplus \left((\mathring{A}^d)_{\mathbf{1}} \otimes_{\mathbb{K}} \mathfrak{g} \right) \quad \forall d \in \mathbb{N}_+$$

this in turn is a decreasing filtration of Lie subalgebras of $\mathfrak{n}_{\mathfrak{g}}(A)$, with $\mathfrak{n}_{\mathfrak{g}}^{(N)}(A) = 0$ for $N \gg 0$.

As a matter of notation, let us consider the case of an element $\eta \otimes Y \in \mathfrak{n}_{\mathfrak{g}}(A)$ with $\eta \in A_{\mathbf{1}}$, $Y \in \mathfrak{g}_{\mathbf{1}}$. By definition $\eta^2 = 0$, hence if we express $\exp(\eta Y)$ as a formal series we actually have $\exp(\eta Y) = 1 + \eta Y$. Similarly, for every $cX = c \otimes X \in \mathfrak{n}_{\mathfrak{g}}(A)$ with $c \in A_{\mathbf{0}}$, $X \in \mathfrak{g}_{\mathbf{0}}$ if $c^2 = 0$ then also the formal series expression of $\exp(cX)$ reads $\exp(cX) = 1 + cX$.

For later use, we fix a \mathbb{K} -basis B of \mathfrak{g} of the form $B := B_{\mathbf{0}} \sqcup B_{\mathbf{1}}$ with $B_{\mathbf{0}} = \{X_j\}_{j \in J}$, resp. $B_{\mathbf{1}} = \{Y_i\}_{i \in I}$, being a \mathbb{K} -basis of $\mathfrak{g}_{\mathbf{0}}$, resp. of $\mathfrak{g}_{\mathbf{1}}$. Moreover, we fix any total order \leq on both I and J , so that both $B_{\mathbf{0}}$ and $B_{\mathbf{1}}$ are totally ordered and, declaring elements from $B_{\mathbf{0}}$ to be less than those of $B_{\mathbf{1}}$ — in a nutshell, setting $B_{\mathbf{0}} \leq B_{\mathbf{1}}$ — overall the whole basis B is totally ordered too.

Our next goal is to find yet another description of the subgroup $N_G(A) = \exp(\mathfrak{n}_{\mathfrak{g}}(A))$. We need for this an auxiliary result:

Lemma 4.2.2. *Let $S_1, \dots, S_{\ell} \in \mathfrak{n}_{\mathfrak{g}}(A)$ with $S_i \in \mathfrak{n}_{\mathfrak{g}}^{(d_i)}(A)$ for some $d_i \in \mathbb{N}_+$ ($i = 1, \dots, \ell$). Then there exist $T_1, \dots, T_k \in \mathfrak{n}_{\mathfrak{g}}(A)$ such that $T_j \in \mathfrak{n}_{\mathfrak{g}}^{(\partial_j)}(A)$ with $\partial_j \geq d_{a_j} + d_{b_j}$ for some $a_j, b_j \in \{1, \dots, \ell\}$ (for all $j = 1, \dots, k$), and*

$$\exp(S_1 + \dots + S_{\ell}) = \exp(S_1) \cdots \exp(S_{\ell}) \exp(T_1) \cdots \exp(T_k)$$

Proof. Writing all exponentials as formal series (actually *finite sum*, because of the nilpotency of all elements in $\mathfrak{n}_{\mathfrak{g}}(A)$, cf. §4.1.8), the claim follows at once from definitions by induction on ℓ via a straightforward application of Baker-Campbell-Hausdorff formula. \square

We can now provide our new description of the subgroup $N_G(A) = \exp(\mathfrak{n}_{\mathfrak{g}}(A))$:

Proposition 4.2.3. *The subgroup $N_G(A) = \exp(\mathfrak{n}_{\mathfrak{g}}(A))$ of $G(A)$ is generated by the set*

$$\Gamma_B := \left\{ \exp(t_j X_j), \exp(\eta_i Y_i) \mid t_j \in \mathring{A}_{\mathbf{0}}, \eta_i \in \mathring{A}_{\mathbf{1}} = A_{\mathbf{1}}, \forall j \in J, i \in I \right\}$$

where $\{X_j\}_{j \in J} \sqcup \{Y_i\}_{i \in I} = B_{\mathbf{0}} \sqcup B_{\mathbf{1}} = B$ is the \mathbb{K} -basis of \mathfrak{g} chosen in §4.2.1 above.

Proof. Let $n \in N_G(A) = \exp(\mathfrak{n}_{\mathfrak{g}}(A))$, say $n = \exp(Z)$ with $Z \in \mathfrak{n}_{\mathfrak{g}}(A)$; clearly we can assume $Z \neq 0$. Using our fixed, ordered, \mathbb{K} -basis B of \mathfrak{g} our Z expands into $Z = \sum_{j \in J} t'_j X_j + \sum_{i \in I} \eta'_i Y_i$ for some $t'_j \in \mathring{A}_{\mathbf{0}}$ and $\eta'_i \in \mathring{A}_{\mathbf{1}}$, by the very definition of $\mathfrak{n}_{\mathfrak{g}}(A)$. By Lemma 4.2.2, this implies that

$$\exp(Z) = \exp\left(\sum_{j \in J} t'_j X_j + \sum_{i \in I} \eta'_i Y_i\right) = \overrightarrow{\prod}_{j \in J} \exp(t'_j X_j) \overrightarrow{\prod}_{i \in I} \exp(\eta'_i Y_i) \cdot \exp(Z_1^{(1)}) \cdots \exp(Z_{k_1}^{(1)})$$

for some $Z_1^{(1)}, \dots, Z_{k_1}^{(1)} \in \mathfrak{n}_{\mathfrak{g}}(A)$, where the symbols $\overrightarrow{\prod}_{j \in J}$ and $\overrightarrow{\prod}_{i \in I}$ denotes ordered products.

Even more, the Lemma ensures that the new terms $Z_h^{(1)}$'s actually “lie deeper”, in the decreasing filtration of $\mathfrak{n}_{\mathfrak{g}}(A)$ given by the $\mathfrak{n}_{\mathfrak{g}}^{(d)}(A)$'s, than the initial Z we started with, and this allows us to get to the end iterating this argument, in finitely many steps.

Indeed, formally speaking we define $d(T) \in \mathbb{N}+$ by the condition $T \in \mathfrak{n}_{\mathfrak{g}}^{(d(T))}(A) \setminus \mathfrak{n}_{\mathfrak{g}}^{(d(T)+1)}(A)$. Then in our construction Lemma 4.2.2 ensures that for the newly occurring elements $Z_1^{(1)}, \dots, Z_{k_1}^{(1)}$ we have $d(Z_h^{(1)}) > d(Z)$ for all $h = 1, \dots, k$. Now we can repeat our argument, with the $Z_h^{(1)}$'s playing the role of Z , and find similar results, i.e. a new expression for $\exp(Z)$ of the form

$$\exp(Z) = \prod_{j \in J} \exp(t'_j X_j) \prod_{i \in I} \exp(\eta'_i Y_i) \cdot \prod_{s=1}^{k_1} \left(\prod_{j \in J} \exp(t''_{s,j} X_j) \prod_{i \in I} \exp(\eta''_{s,i} Y_i) \right) \cdot \prod_{r=1}^{k_2} \exp(Z_r^{(2)})$$

for some $Z_1^{(2)}, \dots, Z_{k_2}^{(2)} \in \mathfrak{n}_{\mathfrak{g}}(A)$ such that $d(Z_r^{(2)}) > \min \{d(Z_h^{(1)})\}_{h=1, \dots, k_1}$ for all $r = 1, \dots, k_2$. Iterating this process we find at each step new factors belonging to the set Γ_B and possibly new factors of the form $\exp(Z_q^{(c)})$, $q = 1, \dots, k_c$, such that the sequence of the natural numbers $n_c := \min \{d(Z_q^{(c)})\}_{q=1, \dots, k_c}$ is strictly increasing. Then, since we have $\mathfrak{n}_{\mathfrak{g}}^{(N)}(A) = \{0\}$ for $N \gg 0$, after finitely many steps we necessarily find $Z_q^{(c)} \equiv 0$, i.e. no further terms actually occur, and $n = \exp(Z)$ is eventually written as a product of elements in Γ_B , thus the latter is indeed a generating set for $N_G(A) = \exp(\mathfrak{n}_{\mathfrak{g}}(A))$, as claimed. \square

Before going on, let us consider elements in $G(A)$ of the form $\exp(tX)$ or $\exp(\eta Y)$ — with $t \in A_0$ such that $t^2 = 0$ (so that $t \in \mathring{A}_0$ indeed), $X \in \mathfrak{g}_0$, $\eta \in \mathring{A}_1$, $Y \in \mathfrak{g}_1$ — like those (see above) that generate $\exp(\mathfrak{n}_{\mathfrak{g}}(A)) = N_G(A)$. Since both t and η have square zero, the formal power series expansion of both $\exp(tX)$ and $\exp(\eta Y)$ is actually truncated at first order, i.e. it reads $\exp(tX) = (1 + tX)$ and $\exp(\eta Y) = (1 + \eta Y)$ respectively. Using this more inspiring notation, we now consider some interesting relations inside $G(A)$ involving group elements of this type.

Lemma 4.2.4. *Let $A \in (\mathbf{Wsalg})_{\mathbb{K}}$, $\eta, \eta', \eta'' \in A_1$, $Y, Y' \in \mathfrak{g}_1$, $X \in \mathfrak{g}_0$ and $g_0 \in G_0(A)$. Then inside $G(A)$ we have (notation of Definition 2.2.1)*

- (a) $(1 + \eta \eta' [Y, Y']) = \exp(\eta \eta' [Y, Y']) \in G_0(A)$;
- (b) $(1 + \eta Y) g_0 = g_0 (1 + \eta \text{Ad}(g_0^{-1})(Y))$;
- (c) $(1 + \eta' Y') (1 + \eta'' Y'') = (1 + \eta'' \eta' [Y', Y'']) (1 + \eta'' Y'') (1 + \eta' Y')$;
- (d) $(1 + \eta Y') (1 + \eta Y'') = (1 + \eta(Y' + Y'')) = (1 + \eta Y'') (1 + \eta Y')$;
- (e) $(1 + \eta' Y) (1 + \eta'' Y) = (1 + \eta'' \eta' Y^{(2)}) (1 + (\eta' + \eta'') Y)$;
- (f) $(1 + \eta Y) (1 + \eta' \eta'' X) = (1 + \eta' \eta'' X) (1 + \eta \eta' \eta'' [Y, X]) (1 + \eta Y) =$
 $= (1 + \eta' \eta'' X) (1 + \eta Y) (1 + \eta \eta' \eta'' [Y, X])$.
- (g) Let $(h, k) := h k h^{-1} k^{-1}$ be the commutator of elements h and k in a group. Then
 $((1 + \eta Y), (1 + \eta' Y')) = (1 + \eta' \eta [Y, Y'])$, $((1 + \eta Y), (1 + \eta Y')) = (1 + \eta(Y + Y'))$
 $((1 + \eta' Y), (1 + \eta'' Y)) = (1 + \eta'' \eta' Y^{(2)})^2 = (1 + \eta'' \eta' 2 Y^{(2)}) = (1 + \eta'' \eta' [Y, Y])$
(N.B.: taking the rightmost term in the last identity, the latter is a special case of the first).

Proof. Writing all exponentials as formal power series (actually *finite sum*, as noticed above), the claim follows at once from definitions via straightforward applications of the Baker-Campbell-Hausdorff formula. In particular, for claim (a) we have that $(1 + \eta \eta' [Y, Y'])$ is just the formal power series expansion of $\exp(\eta \eta' [Y, Y'])$, and the latter belong to $\exp(\mathring{A}_0 \mathfrak{g}_0) \subseteq G_0(A)$. \square

We still need to introduce some auxiliary objects associated with G :

Definition 4.2.5. Let G be a Lie supergroup, as above. For any $A \in (\mathbf{Wsalg})_{\mathbb{K}}$, we define:

$$(a) \quad G^-(A) := \left\{ \prod_{s=1}^n (1 + \eta_s Y_s) \mid n \in \mathbb{N}, (\eta_s, Y_s) \in A_1 \times \mathfrak{g}_1 \ \forall s \in \{1, \dots, n\} \right\} \quad (\subseteq G(A))$$

$$(b) \quad N_G^{[2]}(A) := \exp\left(A_1^{[2]} \otimes_{\mathbb{K}} [\mathfrak{g}_1, \mathfrak{g}_1]\right) \quad (\subseteq N_{G_0}(A_0) = N_G(A) \cap G_0(A))$$

(c) for any fixed \mathbb{K} -basis $\{Y_i\}_{i \in I}$ of \mathfrak{g}_1 (for some index set I) and any fixed total order in I ,

$$G_-^{\leq}(A) := \left\{ \overrightarrow{\prod}_{i \in I} (1 + \eta_i Y_i) \mid \eta_i \in A_1 \ \forall i \in I \right\} \quad (\subseteq G^-(A) \subseteq G_p(A))$$

where $\overrightarrow{\prod}_{i \in I}$ denotes an *ordered product* — with respect to the fixed total order in I . \diamond

From now on, then, we fix a \mathbb{K} -basis $\{Y_i\}_{i \in I}$ of \mathfrak{g}_1 (for some index set I) and we fix in I a total order, as required in Definition 4.2.5(c) above.

With our first step we start finding some new, interesting “factorization results” for $G(A)$:

Proposition 4.2.6. *Let G be a Lie supergroup as above, let $\{Y_i\}_{i \in I}$ be a totally ordered \mathbb{K} -basis of \mathfrak{g}_1 — for some total order in the set I , and let $A \in (\mathbf{Wsalg})_{\mathbb{K}}$ be any Weil superalgebra. Then:*

- (a) $G^-(A)$ is nothing but the subgroup of $G_p(A)$ generated by $G_-^{\leq}(A)$;
- (b) there exist group-theoretic factorizations

$$G^-(A) = N_G^{[2]}(A) \cdot G_-^{\leq}(A) \quad , \quad G^-(A) = G_-^{\leq}(A) \cdot N_G^{[2]}(A) \quad (4.10)$$

$$N_G(A) = N_{G_0}(A_0) \cdot G_-^{\leq}(A) \quad , \quad N_G(A) = G_-^{\leq}(A) \cdot N_{G_0}(A_0) \quad (4.11)$$

$$G(A) = G_0(A) \cdot G_-^{\leq}(A) \quad , \quad G(A) = G_-^{\leq}(A) \cdot G_0(A) \quad (4.12)$$

Proof. (a) Let us denote by $\langle G_-^{\leq}(A) \rangle$ the subgroup of $G(A)$ generated by $G_-^{\leq}(A)$. It is clear by definition that $G^-(A)$ is the subgroup in $G(A)$ generated by $\{(1 + \eta Y) \mid \eta \in A_1, Y \in \mathfrak{g}_1\}$: thus it is enough to prove that each of its generators $(1 + \eta Y)$ actually belongs to $\langle G_-^{\leq}(A) \rangle$.

Now, given $Y \in \mathfrak{g}_1$ let $Y = \sum_{i \in I} c_i Y_i$ (with $c_i \in \mathbb{K}$) be its \mathbb{K} -linear expansion with respect to the \mathbb{K} -basis $\{Y_i\}_{i \in I}$ of \mathfrak{g}_1 . Then repeated applications of the identity in Lemma 4.2.4(d) yield $(1 + \eta Y) = (1 + \eta \sum_{i \in I} c_i Y_i) = (1 + \sum_{i \in I} (c_i \eta) Y_i) = \overrightarrow{\prod}_{i \in I} (1 + (c_i \eta) Y_i) \in G_-^{\leq}(A)$, q.e.d.

(b) We begin with the proof of (4.10): by left-right symmetry, it is enough to prove the left-hand side, that is $G^-(A) = N_G^{[2]}(A) \cdot G_-^{\leq}(A)$, so we focus on that. By claim (a), any element of $G^-(A)$ can be written as a (unordered) product of the form $\prod_{k=1}^N (1 + \eta_k Y_{i_k})$ with $\eta_k \in A_1$ and $i_k \in I$ for all k . Our goal is to prove the following

Claim: Any (unordered) product of the form $\prod_{k=1}^N (1 + \eta_k Y_{i_k})$ can be “re-ordered”, namely it can be re-written as an element of $N_G^{[2]}(A) \cdot G_-^{\leq}(A)$.

To prove this *Claim*, let \mathfrak{a} be the (two-sided) ideal of A generated by the η_k ’s, and denote by \mathfrak{a}^n its n -th power ($n \in \mathbb{N}$); as the η_k ’s are N elements and are odd, we have $\mathfrak{a}^n = \{0\}$ for $n > N$.

Let us denote by \leq our fixed total order in I . Given the product $\prod_{k=1}^N (1 + \eta_k Y_{i_k})$, we define its *inversion number* as being the number of occurrences of two consecutive indices k_s and k_{s+1} for which $i_{k_s} \not\leq i_{k_{s+1}}$: then the product itself is *ordered* if and only if its inversion number is zero.

Now assume the product $g := \prod_{k=1}^N (1 + \eta_k Y_{i_k})$ is unordered: then there exists at least an inversion, say $i_{k_s} \not\leq i_{k_{s+1}}$, i.e. either $i_{k_s} > i_{k_{s+1}}$ or $i_{k_s} = i_{k_{s+1}}$. Once again, using some of the

relations considered in Lemma 4.2.4 we can re-write the product of these two “unordered factors” $(1 + \eta_{k_s} Y_{i_{k_s}}) (1 + \eta_{k_{s+1}} Y_{i_{k_{s+1}}})$ in either form (depending on whether $i_{k_s} > i_{k_{s+1}}$ or $i_{k_s} = i_{k_{s+1}}$)

$$\begin{aligned} (1 + \eta_{k_s} Y_{i_{k_s}}) (1 + \eta_{k_{s+1}} Y_{i_{k_{s+1}}}) &= (1 + \eta_{k_{s+1}} \eta_{k_s} [Y_{i_{k_{s+1}}}, Y_{i_{k_s}}]) (1 + \eta_{k_{s+1}} Y_{i_{k_{s+1}}}) (1 + \eta_{k_s} Y_{i_{k_s}}) \\ (1 + \eta_{k_s} Y_{i_{k_s}}) (1 + \eta_{k_{s+1}} Y_{i_{k_s}}) &= (1 + \eta_{k_{s+1}} \eta_{k_s} Y_{i_{k_s}}^{(2)}) (1 + (\eta_{k_s} + \eta_{k_{s+1}}) Y_{i_{k_s}}) \end{aligned}$$

according to whether $i_{k_s} > i_{k_{s+1}}$ or $i_{k_s} = i_{k_{s+1}}$ respectively. In either case, re-writing in this way the product of the k_s -th and the k_{s+1} -th factor in the original product $g := \prod_{k=1}^N (1 + \eta_k Y_{i_k})$, we end up with another product expression where we did eliminate one inversion, but we “payed the price” of inserting a *new factor*. However, in both cases the newly added factor is of the form $(1 + a X)$ for some $X \in [\mathfrak{g}_1, \mathfrak{g}_1]$ and $a \in \mathfrak{a}^2$, so that $(1 + a X) \in N_G^{[2]}(A)$.

By repeated use of relations of the form $(1 + \eta Y) \cdot g_0 = g_0 \cdot (1 + \eta \text{Ad}(g_0^{-1})(Y))$ we can shift the newly added factor $(1 + a X)$ to the leftmost position in g — now re-written once more in yet a different product form — at the cost of inserting several *new factors of the form* $(1 + b_t Z_t)$ for some $Z_t \in \mathfrak{g}_1$ and $b_t \in \mathfrak{a}^3$. Moreover, by repeated use of relations of the form $(1 + \eta Y') \cdot (1 + \eta Y'') = (1 + \eta (Y' + Y''))$ we can re-write each of these new factors as a product of factors of the form $(1 + \eta'_h Y_{i'_h})$ where $\eta'_h \in A_1$ is a multiple of some b_t , so that $\eta'_h \in \mathfrak{a}^3$ too.

Eventually, we find a new factorization of the original element $g := \prod_{k=1}^N (1 + \eta_k Y_{i_k})$ in the new form $g := g'_0 \cdot \prod_{h=1}^{N'} (1 + \eta'_h Y_{i'_h})$, where now $g'_0 \in N_G^{[2]}(A)$ and the factors $(1 + \eta'_h Y_{i'_h})$ satisfy the following conditions:

- (a) each factor $(1 + \eta'_h Y_{i'_h})$ is either one of the old factors $(1 + \eta_k Y_{i_k})$ or a truly new factor;
- (b) in every (truly) new factor $(1 + \eta'_h Y_{i'_h})$ one has $\eta'_h \in \mathfrak{a}^3$;
- (c) the number of inversions among factors $(1 + \eta'_h Y_{i'_h}) = (1 + \eta_k Y_{i_k})$ of the old type is one less than before.

Iterating this procedure, after finitely many steps we obtain a new factorization of the initial element $g := \prod_{k=1}^N (1 + \eta_k Y_{i_k})$ of the form $g = g''_0 \cdot \prod_{h=1}^{N''} (1 + \eta''_h Y_{i''_h})$ where $g''_0 \in N_G^{[2]}(A)$ and the factors $(1 + \eta''_h Y_{i''_h})$ enjoy properties (a) and (b) above plus the “optimal version” of (c), namely

- (c+) the number of inversions among factors of the old type is zero.

Now we apply the same “reordering operation” to the product $\prod_{h=1}^{N''} (1 + \eta''_h Y_{i''_h})$. By construction, an inversion now can occur only among two factors of new type or among an old and a new factor; but then the two coefficients η''_h involved by this inversion belong to \mathfrak{a} and at least one of them is in \mathfrak{a}^3 . It follows that when one performs the “reordering operation” onto the pair of factors involved in the inversion the new factor which pops up necessarily involves a coefficient in \mathfrak{a}^4 . As this applies for any possible inversion, in the end we shall find a new factorization of g of the form

$$g = g''_0 \cdot \hat{g}_0 \cdot \prod_{t=1}^{\hat{N}} (1 + \hat{\eta}_t Y_{\hat{i}_t})$$

in which $\hat{g}_0 \in G_+(A)$ and the factors $(1 + \hat{\eta}_t Y_{\hat{i}_t})$ are either old factors $(1 + \eta_k Y_{i_k})$, with no inversions among them, or new factors such that $\hat{\eta}_t \in \mathfrak{a}^5$.

In order to conclude, we can iterate at will this procedure: then — as $\mathfrak{a}^n = \{0\}$ for $n > N$ — after finitely many steps we shall no longer find any new factor coming in; therefore, we eventually find a last factorization of g of the form

$$g = \tilde{g}_0 \cdot \prod_{\ell=1}^{\tilde{N}} (1 + \tilde{\eta}_\ell Y_{\tilde{i}_\ell}) = \tilde{g}_0 \cdot \vec{\prod}_{\ell=1}^{\tilde{N}} (1 + \tilde{\eta}_\ell Y_{\tilde{i}_\ell})$$

in which $\tilde{g}_0 \in N_G^{[2]}(A)$ and $\prod_{\ell=1}^{\tilde{N}} (1 + \tilde{\eta}_\ell Y_{i_\ell}) = \overrightarrow{\prod}_{\ell=1}^{\tilde{N}} (1 + \tilde{\eta}_\ell Y_{i_\ell}) \in G_-^\leq(A)$ is an ordered product, as required: this means exactly $g \in N_G^{[2]}(A) \cdot G_-^\leq(A)$, thus the *Claim* is proved.

Our *Claim* above ensures that $G^-(A) \subseteq N_G^{[2]}(A) \cdot G_-^\leq(A)$. Now we have to prove the converse inclusion. To this end, recall that $N_G^{[2]}(A)$ is generated by elements of the form $(1 + cX)$ with $c \in A_1^{[2]}$ and $X \in [\mathfrak{g}_1, \mathfrak{g}_1]$ — still with notation of type $(1 + cX) := \exp(cX)$, as before — therefore $c = \sum_s \alpha'_s \alpha''_s$ and $X = \sum_r [Y'_r, Y''_r]$ for some $\alpha'_s, \alpha''_s \in A_1$ and some $Y'_r, Y''_r \in \mathfrak{g}_1$. Now, inside $G_0(A_0)$ we always have relations of the form

$$(1 + a_1 Z) \cdot (1 + a_2 Z) = (1 + (a_1 + a_2) Z) = (1 + a_2 Z) \cdot (1 + a_1 Z)$$

for all $Z \in \mathfrak{g}_0$ and $a_1, a_2 \in A_0$ such that $a_1^2 = 0 = a_2^2$. Applying this repeatedly to $(1 + cX) = (1 + (\sum_s \alpha'_s \alpha''_s) X)$ yields

$$(1 + cX) = (1 + (\sum_s \alpha'_s \alpha''_s) X) = \prod_s (1 + \alpha'_s \alpha''_s X) \quad (4.13)$$

where the factors in the final product can be taken in any order, as they mutually commute.

Now recall instead that $X = \sum_r [Y'_r, Y''_r]$, hence each factor $(1 + \alpha'_s \alpha''_s X)$ in (4.13) reads

$$(1 + \alpha'_s \alpha''_s X) = (1 + \sum_r \alpha'_s \alpha''_s [Y'_r, Y''_r]) \quad (4.14)$$

In addition, inside $G_0(A_0)$ we also have, for all $Z_1, Z_2 \in \mathfrak{g}_0$ and $a \in A_0$ such that $a^2 = 0$, relations of the form

$$(1 + a Z_1) \cdot (1 + a Z_2) = (1 + a (Z_1 + Z_2)) = (1 + a Z_2) \cdot (1 + a Z_1)$$

With repeated applications of this to the right-hand side term in (4.14) above we eventually get

$$(1 + \alpha'_s \alpha''_s X) = (1 + \sum_r \alpha'_s \alpha''_s [Y'_r, Y''_r]) = \prod_r (1 + \alpha'_s \alpha''_s [Y'_r, Y''_r]) \quad (4.15)$$

As next step, recall that in $G(A)$ also hold relations of the form

$$(1 + \eta'' Y'') \cdot (1 + \eta' Y') = (1 + \eta' \eta'' [Y', Y'']) \cdot (1 + \eta' Y') \cdot (1 + \eta'' Y'')$$

that we can re-shape as

$$(1 + \eta' \eta'' [Y', Y'']) = ((1 + \eta'' Y''), (1 + \eta' Y')) \quad (4.16)$$

where in right-hand side we used standard group-theoretical notation $(a, b) := a b a^{-1} b^{-1}$ for the commutator of any two elements a and b in a given group. Now (4.16) together with (4.15) gives

$$(1 + \alpha'_s \alpha''_s X) = \prod_r (1 + \alpha'_s \alpha''_s [Y'_r, Y''_r]) = \prod_r ((1 + \alpha''_s Y''_r), (1 + \alpha'_s Y'_r)) \quad (4.17)$$

Eventually, matching (4.17) with (4.13) we get

$$(1 + cX) = \prod_s (1 + \alpha'_s \alpha''_s X) = \prod_{s,r} ((1 + \alpha''_s Y''_r), (1 + \alpha'_s Y'_r)) \in \langle G_-^\leq(A) \rangle = G^-(A)$$

(where the factors in the last product can be taken in any order). Thus the element $(1 + cX)$ belongs to $G^-(A)$; since $N_G^{[2]}(A)$ is generated by such elements, we get $N_G^{[2]}(A) \subseteq G^-(A)$, whence clearly $N_G^{[2]}(A) \cdot G_-^\leq(A) \subseteq G^-(A)$ and we are done.

Now we go and prove (4.11): like before, it is enough to prove the left-hand side, that is $N_G(A) = N_{G_0}(A_0) \cdot G^{\leq}(A)$, by left-right symmetry.

Thanks to Proposition 4.2.3, we can take as generators of $N_G(A)$ the elements of the set

$$\left\{ \exp(t_j X_j), \exp(\eta_i Y_i) = (1 + \eta_i Y_i) \mid t_j \in \mathring{A}_0, \eta_i \in \mathring{A}_1 = A_1, \forall j \in J, i \in I \right\} \quad (4.18)$$

where $\{X_j\}_{j \in J}$ is any \mathbb{K} -basis of \mathfrak{g}_0 and $\{Y_i\}_{i \in I}$ is our fixed, totally ordered \mathbb{K} -basis of \mathfrak{g}_1 . Therefore, our aim is to prove that any $n \in N_G(A)$, originally expressed as an unordered product of factors taken from (4.18), can be “re-ordered” so to read as an *ordered product* of the form $n_0 \prod_{i \in I}^{\rightarrow} (1 + \hat{\eta}_i Y_i)$, for some $n_0 \in N_{G_0}(A_0)$ and $\hat{\eta}_i \in A_1$, which does belong to $N_{G_0}(A_0) \cdot G^{\leq}(A)$.

First of all, whenever in our original product n we have two consecutive factors $(1 + \eta_s Y_{i_s})$ and $n'_0 := \exp(t_j X_j) \in N_{G_0}(A_0) (\subseteq G_0(A_0))$ — that is, we have a “sub-product” of the form $(1 + \eta_s Y_{i_s}) \cdot n'_0$ — using relations (b) and (d) in Lemma 4.2.4 we can re-write this subproduct as

$$(1 + \eta_s Y_{i_s}) n'_0 = n'_0 \left(1 + \eta_s \text{Ad}((n'_0)^{-1})(Y_{i_s}) \right) = n'_0 \left(1 + \eta_s \sum_{j \in I} c_{s,j} Y_j \right) = n'_0 \prod_{j \in I}^{\rightarrow} (1 + \eta_s c_{s,j} Y_j) \quad (4.19)$$

for suitable $c_{s,j} \in \mathbb{K}$ ($j \in I$); note in particular that the rightmost term in the chain of identities (4.19) does belong to $N_{G_0}(A_0) \cdot G^{\leq}(A)$, i.e. it has the form we are looking for. Applying this procedure, whenever in our element n — written as a product as above — we have a factor of type $n'_0 \in N_{G_0}(A_0)$ that occurs on the right of any factor of type $(1 + \eta_s Y_{i_s})$, we can “move the former to the left of the latter” in the sense that we apply (4.19). Then, after finitely many repetitions of this move we end up with a new factorization of n of the form

$$n = n''_0 \cdot \prod_{s=1}^N \prod_{j \in I}^{\rightarrow} (1 + \eta_s k_{s,j} Y_j) \quad (4.20)$$

for some $n''_0 \in N_{G_0}(A_0)$ and $k_{s,j} \in \mathbb{K}$ (where N is the number of factors of type $(1 + \eta_s Y_{i_s})$ occurring in the initial factorization of n). Now, in this last factorization, the right-hand side gives

$$\prod_{s=1}^N \prod_{j \in I}^{\rightarrow} (1 + \eta_s k_{s,j} Y_j) \in G^-(A) = N_G^{[2]}(A) \cdot G^{\leq}(A)$$

thanks to (4.10). This together with (4.20) gives

$$n \in N_{G_0}(A_0) \cdot G^-(A) = N_{G_0}(A_0) \cdot N_G^{[2]}(A) \cdot G^{\leq}(A) = N_{G_0}(A_0) \cdot G^{\leq}(A)$$

— because $N_G^{[2]}(A)$ is a subgroup of $N_{G_0}(A_0)$, by construction — and we are done.

Finally, as to (4.12) we prove it via the following chain of identities:

$$\begin{aligned} G(A) &= G_0(\mathbb{K}) \cdot N_G(A) = G_0(\mathbb{K}) \cdot (N_{G_0}(A_0) \cdot G^{\leq}(A)) = \\ &= (G_0(\mathbb{K}) \cdot N_{G_0}(A_0)) \cdot G^{\leq}(A) = G_0(A) \cdot G^{\leq}(A) \end{aligned}$$

where we first used Borek’s splitting for $G(A)$ — cf. (4.1) — and then (4.11). \square

The previous proposition provides remarkable factorizations for the A -points of the Lie supergroup G and their remarkable subgroups N_G and G^- . Our ultimate goal is to improve such a result, eventually achieving stronger factorization results: in case of G itself, this will be what is (more or less) known as “global splitting” for Lie supergroups. We still need two technical lemmas:

Lemma 4.2.7. *Given a Lie supergroup G and $A \in (\mathbf{Wsalg})_{\mathbb{K}}$, let $\hat{\eta}_i, \check{\eta}_i \in A_1$ and let \mathfrak{a} be an ideal of A such that $\hat{\eta}_i, \check{\eta}_i \in \mathfrak{a}$ and $\alpha_i := \hat{\eta}_i - \check{\eta}_i \in \mathfrak{a}^n$ ($i \in I$) for some $n \in \mathbb{N}_+$. Then*

$$\overrightarrow{\prod}_{i \in I} (1 + [\hat{\eta}_i]_{n+1} Y_i) \cdot \overleftarrow{\prod}_{i \in I} (1 - [\check{\eta}_i]_{n+1} Y_i) = \overrightarrow{\prod}_{i \in I} (1 + [\alpha_i]_{n+1} Y_i) \in G_{\leq}^-(A/\mathfrak{a}^{n+1})$$

where $\overrightarrow{\prod}_{i \in I}$ and $\overleftarrow{\prod}_{i \in I}$ respectively denote an ordered and a reversely-ordered product (with respect to the given order in I) and $[a]_{n+1} \in A/\mathfrak{a}^{n+1}$ stands for the coset modulo \mathfrak{a}^{n+1} of any $a \in A$.

Proof. This is an easy, straightforward consequence of the relations in $G(A)$ listed in Lemma 4.2.4, also taking into account that the assumptions imply, for all $i \in I$,

$$\hat{\eta}_i \check{\eta}_i = \hat{\eta}_i (\hat{\eta}_i - \alpha_i) = \hat{\eta}_i^2 - \hat{\eta}_i \alpha_i = -\hat{\eta}_i \alpha_i \in \mathfrak{a}^{n+1}.$$

□

Lemma 4.2.8. *Given a Lie supergroup G and $A \in (\mathbf{Wsalg})_{\mathbb{K}}$, let $\zeta_i \in A_1$ ($i \in I$) be such that*

$$g := \overrightarrow{\prod}_{i \in I} (1 + \zeta_i Y_i) \in G_0(A) \cap G_{\leq}^-(A).$$

Then $\zeta_i = 0$ for all $i \in I$.

Proof. Recall that, by definition, we have $G(A) := \coprod_{x \in |G|} \text{Hom}_{(\mathbf{salg})_{\mathbb{K}}}(\mathcal{O}_{|G|,x}, A)$; therefore, it makes sense to formally expand the product defining g as

$$g := \overrightarrow{\prod}_{i \in I} (1 + \zeta_i Y_i) = 1 + \sum_{i \in I} \zeta_i Y_i + \mathcal{O}(2) \quad (4.21)$$

where $\mathcal{O}(2)$ is a short-hand notation for “additional summands of higher order in the ζ_i ’s”. Let $\mathfrak{a} := (\{\zeta_i\}_{i \in I})$ be the ideal of A generated by the ζ_i ’s; then (4.21) yields

$$[g]_2 := 1 + \sum_{i \in I} [\zeta_i]_2 Y_i \quad (4.22)$$

inside $G(A/\mathfrak{a}^2) := \coprod_{x \in |G|} \text{Hom}_{(\mathbf{salg})_{\mathbb{K}}}(\mathcal{O}_{|G|,x}, A/\mathfrak{a}^2)$. On the other hand, the assumption $g \in G_0(A) \cap G_{\leq}^-(A)$ implies $[g]_2 \in G_0(A/\mathfrak{a}^2) \cap G_{\leq}^-(A/\mathfrak{a}^2)$ as well, hence in particular — thinking of $[g]_2$ as an A/\mathfrak{a}^2 -valued map — we have $\text{Im}([g]_2) \subseteq (A/\mathfrak{a}^2)_0$.

Now, as $\{Y_i\}_{i \in I}$ is a \mathbb{K} -basis of \mathfrak{g}_1 , there exists a local system of coordinates around the unit point $1_G \in |G|$, say $\{y_i\}_{i \in I}$, such that $Y_i(y_j) = \delta_{i,j}$ for all $i, j \in I$. Then (4.22) gives $[g]_2(y_j) := 1 + \sum_{i \in I} [\zeta_i]_2 Y_i(y_j) = [\zeta_j]_2$, in particular $[g]_2(y_j) = [\zeta_j]_2 \in (A/\mathfrak{a}^2)_1$; but this together with $\text{Im}([g]_2) \subseteq (A/\mathfrak{a}^2)_0$ implies $[\zeta_j]_2 = [0]_2 \in A/\mathfrak{a}^2$, that is $\zeta_j \in \mathfrak{a}^2 = (\{\zeta_i\}_{i \in I})^2$, for all $j \in I$: it follows that $\zeta_j \in \mathfrak{a}^n$ for all $n \in \mathbb{N}$ (and $j \in I$). By construction we have also $\mathfrak{a}^n = 0$ for $n \gg 0$, thus we end up with $\zeta_j = 0$ for all $j \in I$, q.e.d. □

Finally, we can now state and prove the main result of the present subsection:

Theorem 4.2.9. *(existence of Global Splittings for Lie supergroups)*

Let G be a Lie supergroup, and \mathfrak{g} its tangent Lie superalgebra.

(a) *The restriction of group multiplication in G provides isomorphisms of (set-valued) functors*

$$\begin{aligned} N_G^{[2]} \times G_{\leq}^- &\cong G^- & , & & N_{G_0} \times G_{\leq}^- &\cong N_G & , & & G_0 \times G_{\leq}^- &\cong G \\ G_{\leq}^- \times N_G^{[2]} &\cong G^- & , & & G_{\leq}^- \times N_{G_0} &\cong N_G & , & & G_{\leq}^- \times G_0 &\cong G \end{aligned}$$

(b) There exists an isomorphism of (set-valued) functors $\mathbb{A}_{\mathbb{K}}^{0|d-} \cong G_{-}^{\leq}$, with $d_{-} := |I| = \dim_{\mathbb{K}}(\mathfrak{g}_1)$, given on A -points — for every $A \in (\mathbf{Wsalg})_{\mathbb{K}}$ — by

$$\mathbb{A}_{\mathbb{K}}^{0|d-}(A) = A_1^{d-} \longrightarrow G_{-}^{\leq}(A) \quad , \quad (\eta_i)_{i \in I} \mapsto \overrightarrow{\prod}_{i \in I} (1 + \eta_i Y_i)$$

(c) There exist isomorphisms of (set-valued) functors

$$\begin{aligned} N_G^{[2]} \times \mathbb{A}_{\mathbb{K}}^{0|d-} &\cong G^{-} \quad , & N_{G_0} \times \mathbb{A}_{\mathbb{K}}^{0|d-} &\cong N_G \quad , & G_0 \times \mathbb{A}_{\mathbb{K}}^{0|d-} &\cong G \\ \mathbb{A}_{\mathbb{K}}^{0|d-} \times N_G^{[2]} &\cong G^{-} \quad , & \mathbb{A}_{\mathbb{K}}^{0|d-} \times N_{G_0} &\cong N_G \quad , & \mathbb{A}_{\mathbb{K}}^{0|d-} \times G_0 &\cong G \end{aligned}$$

given on A -points — for every $A \in (\mathbf{Wsalg})_{\mathbb{K}}$ — respectively by

$$(g_0, (\eta_i)_{i \in I}) \mapsto g_0 \cdot \overrightarrow{\prod}_{i \in I} (1 + \eta_i Y_i) \quad \text{and} \quad ((\eta_i)_{i \in I}, g_0) \mapsto \overrightarrow{\prod}_{i \in I} (1 + \eta_i Y_i) \cdot g_0$$

Proof. (a) The claim yields a (strong) refinement of the factorization results of Proposition 4.2.6(b), now stated in functorial terms. Just like for that proposition, it is enough to prove half of the results, say those in first line; moreover, we bound ourselves to prove the claim concerning G , as the others go along similarly. In down-to-earth terms, acting pointwise we have to prove that the multiplication map $G_0(A) \times G_{-}^{\leq}(A) \longrightarrow G(A)$ — for any $A \in (\mathbf{Wsalg})_{\mathbb{K}}$ — which is surjective by (4.12) in Proposition 4.2.6(b), is also *injective*: that is, we must show that, if $\hat{g}_+ \hat{g}_- = \check{g}_+ \check{g}_-$ for $\hat{g}_+, \check{g}_+ \in G_0(A)$ and $\hat{g}_-, \check{g}_- \in G_{-}^{\leq}(A)$, then $\hat{g}_+ = \check{g}_+$ and $\hat{g}_- = \check{g}_-$.

From the assumption $\hat{g}_+ \hat{g}_- = \check{g}_+ \check{g}_-$ we get $g := \hat{g}_- \check{g}_-^{-1} = \hat{g}_+^{-1} \check{g}_+ \in G_0(A)$, as $G_0(A)$ is a subgroup in $G(A)$. Now $\hat{g}_- \in G_{-}^{\leq}(A)$ has the form $\hat{g}_- = \overrightarrow{\prod}_{i \in I} (1 + \hat{\eta}_i Y_i)$ and similarly $\check{g}_- = \overleftarrow{\prod}_{i \in I} (1 + \check{\eta}_i Y_i)$ so that $\check{g}_-^{-1} = \overleftarrow{\prod}_{i \in I} (1 - \check{\eta}_i Y_i)$, where once more $\overrightarrow{\prod}$ and $\overleftarrow{\prod}$ respectively denote an ordered and a reversely-ordered product. Therefore we have

$$G_0(A) \ni g := \hat{g}_- \check{g}_-^{-1} = \overrightarrow{\prod}_{i \in I} (1 + \hat{\eta}_i Y_i) \overleftarrow{\prod}_{i \in I} (1 - \check{\eta}_i Y_i) \quad (4.23)$$

Let $\mathfrak{a} := (\{\hat{\eta}_i, \check{\eta}_i\}_{i \in I})$ be the ideal of A generated by the $\hat{\eta}_i$'s and the $\check{\eta}_i$'s; also, for $n \in \mathbb{N}$ write $\pi_n : A \longrightarrow A/\mathfrak{a}^n =: [A]_n$ for the canonical quotient map and $[a]_n := \pi_n(a)$ for every $a \in A$, and then also, correspondingly, $G(\pi_n) : G(A) \longrightarrow G(A/\mathfrak{a}^n) =: G([A]_n)$ for the associated group morphism and $[y]_n := G(\pi_n)(y)$ for every $y \in G(A)$. Now (4.23) along with Lemma 4.2.7 gives

$$[g]_2 = \overrightarrow{\prod}_{i \in I} (1 + [\hat{\eta}_i]_2 Y_i) \overleftarrow{\prod}_{i \in I} (1 - [\check{\eta}_i]_2 Y_i) = \overrightarrow{\prod}_{i \in I} (1 + [\alpha_i]_2 Y_i) \in G_{-}^{\leq}([A]_2)$$

with $\alpha_i := \hat{\eta}_i - \check{\eta}_i \in \mathfrak{a}$ for all i . Since it is also $[g]_2 \in G_0([A]_2)$, we can apply Lemma 4.2.8, with $[A]_2$ playing the rôle of A , thus finding $[\alpha_i]_2 = [0]_2 \in [A]_2$, that is $\alpha_i \in \mathfrak{a}^2$, for all $i \in I$. But now Lemma 4.2.8 applies again, with $[A]_3$ playing the rôle of A , yielding $[\alpha_i]_3 = [0]_3 \in [A]_3$ hence $\alpha_i \in \mathfrak{a}^3$, for all $i \in I$. Then we iterate, finding by induction that $\alpha_i \in \mathfrak{a}^n$ (for $i \in I$) for all $n \in \mathbb{N}_+$; but $\mathfrak{a}^n = \{0\}$ for $n \gg 0$, so $\hat{\eta}_i - \check{\eta}_i =: \alpha_i = 0$, i.e. $\hat{\eta}_i = \check{\eta}_i$, for all $i \in I$. This yields $\hat{g}_- = \check{g}_-$, and from this we get also $\hat{g}_+ = \check{g}_+$, q.e.d.

(b) By definition of G_{-}^{\leq} there exists a functor epimorphism $\Theta : \mathbb{A}_{\mathbb{K}}^{0|d-} \longrightarrow G_{-}^{\leq}$ which is given on every single $A \in (\mathbf{Wsalg})_{\mathbb{K}}$ by

$$\Theta_A : \mathbb{A}_{\mathbb{K}}^{0|d-}(A) := A_1^{\times d-} \longrightarrow G_{-}^{\leq}(A) \quad , \quad (\eta_i)_{i \in I} \mapsto \Theta_A((\eta_i)_{i \in I}) := \overrightarrow{\prod}_{i \in I} (1 + \eta_i Y_i)$$

We prove now that all these Θ_A 's are injective, so that Θ is indeed an isomorphism.

Let $(\hat{\eta}_i)_{i \in I}, (\check{\eta}_i)_{i \in I} \in A_1^{\times d_-}$ be such that $\Theta_A((\hat{\eta}_i)_{i \in I}) = \Theta_A((\check{\eta}_i)_{i \in I})$, in other words we have $\prod_{i \in I} (1 + \hat{\eta}_i Y_i) = \prod_{i \in I} (1 + \check{\eta}_i Y_i)$. Then we can replay the proof of claim (a), now with $\hat{g}_+ := 1 =: \check{g}_+$; the outcome will be again $\hat{\eta}_i = \check{\eta}_i$ for all $i \in I$, that is $(\hat{\eta}_i)_{i \in I} = (\check{\eta}_i)_{i \in I}$, q.e.d.

(c) It follows at once from claims (a) and (b) together. \square

Remark 4.2.10. For every Lie supergroup G , we shall refer to the isomorphisms in claim (a) and/or (c) of Theorem 4.2.9 as to “Global Splittings” of G — or of N_G , or of G^- , respectively. The existence of such splittings is more or less known among specialists, but usually stated (and proved), to the best of the author’s knowledge, in a different manner, in sheaf-theoretic terms.

5 From super Harish-Chandra pairs to Lie supergroups

In this section we provide a quasi-inverse Ψ to the functor Φ — in all its versions: smooth, real analytic and complex holomorphic — of Theorem 3.2.6. The philosophy is that, for any super Harish-Chandra pair \mathcal{P} , we define as associated $\Psi(\mathcal{P}) := G_{\mathcal{P}}$ a suitable functor from Weil superalgebras to groups, and then prove that it has the “right properties”. Concretely, we follow the pattern provided by the *Global Splitting Theorem* for Lie supergroups, which tells us how our would-be Lie supergroup $G_{\mathcal{P}}$ should look like, in terms of \mathcal{P} itself.

5.1 Supergroup functors out of super Harish-Chandra pairs

This subsection is devoted to constructing, starting from a given super Harish-Chandra pair, say $\mathcal{P} = (G_+, \mathfrak{g})$, a convenient supergroup functor, say $G_{\mathcal{P}}$, that we eventually prove (later on) is indeed a Lie supergroup. Our method is as follows: such a $G_{\mathcal{P}}$ is given as a group-valued functor on the category of Weil \mathbb{K} -superalgebras, such that each single group $G_{\mathcal{P}}(A)$ — with $A \in (\mathbf{Wsalg})_{\mathbb{K}}$ — is defined by generators and relations, in a uniform way (with respect to A). The guiding idea that inspires the desired presentation by generators and relations is somewhat simple: as part of our ultimate goal, we want $\Phi(G_{\mathcal{P}}) = \mathcal{P} (= (G_+, \mathfrak{g}))$, thus we must have $(G_{\mathcal{P}})_{\mathbf{0}} = G_+$ and $\text{Lie}(G_{\mathcal{P}}) = \mathfrak{g}$. The former requirement already gives us the reduced Lie subgroup (which somehow holds all the “truly geometrical” content of the supergroup) of $G_{\mathcal{P}}$; the latter requirement instead prescribes what the Lie superalgebra of $G_{\mathcal{P}}$ must be: then we might think of using this to realize the “missing part” of $G_{\mathcal{P}}$ as “ $\exp(\mathfrak{g}_1)$ ”. Thus each group $G_{\mathcal{P}}(A)$ should be presented as generated by $G_+(A) = G_+(A_{\mathbf{0}})$ and $\exp(A_1 \otimes \mathfrak{g}_1)$ and suitable relations.

In order to realize correctly all this, we follow the pattern provided by the analysis of the structure of a Lie supergroup presented in Section 4, in particular the “Global Splitting(s) Theorem” found therein — cf. Theorem 4.2.9 — which essentially prescribes how $G_{\mathcal{P}}$ must be done.

As a matter of notation, hereafter we shall adopt the following: given $\mathcal{P} = (G_+, \mathfrak{g}) \in (\mathbf{sHCp})_{\mathbb{K}}$, $A \in (\mathbf{Wsalg})_{\mathbb{K}}$ and $c \in A_{\mathbf{0}}$ such that $c^2 = 0$, for every $X \in \mathfrak{g}_{\mathbf{0}}$ we set

$$(1_{G_+} + cX) := \exp(cX) \in G_+(A_{\mathbf{0}}) \quad (5.1)$$

which is obviously inspired by the formal series expansion of the “exp” function; when no confusion is possible we shall drop the subscript G_+ and simply write $(1 + cX)$ instead. Similarly, we shall presently introduce new formal elements of type “ $(1 + \eta Y) = \exp(\eta Y)$ ” with $\eta \in A_1$, $Y \in \mathfrak{g}_1$.

Definition 5.1.1. Let $\mathcal{P} := (G_+, \mathfrak{g}) \in (\mathbf{sHCp})_{\mathbb{K}}$ be a super Harish-Chandra pair over \mathbb{K} .

(a) We introduce a functor $G_{\mathcal{P}} : (\mathbf{Wsalg})_{\mathbb{K}} \longrightarrow (\mathbf{group})$ as follows. For any Weil superalgebra $A \in (\mathbf{Wsalg})_{\mathbb{K}}$, we define $G_{\mathcal{P}}(A)$ as being the group with generators the elements of the set

$$\Gamma_A := \{ g_+, (1 + \eta Y) \mid g_+ \in G_+(A), (Y, \eta) \in \mathfrak{g}_1 \times A_1 \} = G_+(A) \cup \{(1 + \eta Y)\}_{(\eta, Y) \in A_1 \times \mathfrak{g}_1}$$

and relations (for $g'_+, g''_+ \in G_+(A)$, $\eta, \eta', \eta'' \in A_1$, $Y, Y', Y'' \in \mathfrak{g}_1$)

$$\begin{aligned} g'_+ \cdot g''_+ &= g'_+ \cdot_{G_+} g''_+ \quad , \quad (1 + \eta Y) \cdot g_+ = g_+ \cdot (1 + \eta \text{Ad}(g_+^{-1})(Y)) \\ (1 + \eta'' Y) \cdot (1 + \eta' Y) &= \left(1_{G_+} + \eta' \eta'' Y^{\langle 2 \rangle} \right)_{G_+} \cdot (1 + (\eta' + \eta'') Y) \\ (1 + \eta'' Y'') \cdot (1 + \eta' Y') &= \left(1_{G_+} + \eta' \eta'' [Y', Y''] \right)_{G_+} \cdot (1 + \eta' Y') \cdot (1 + \eta'' Y'') \\ (1 + \eta Y') \cdot (1 + \eta Y'') &= (1 + \eta (Y' + Y'')) \\ (1 + \eta 0_{\mathfrak{g}_1}) &= 1 \quad , \quad (1 + 0_A Y) = 1 \end{aligned}$$

where the first line just means that for generators chosen in $G_+(A)$ their product, denoted with “ \cdot ”, inside $G_{\mathcal{P}}(A)$ is the same as in $G_+(A)$, where it is denoted with “ \cdot_{G_+} ”; moreover, notation like $\left(1_{G_+} + \eta'_i \eta''_i Y_i^{\langle 2 \rangle} \right)_{G_+}$ and $\left(1_{G_+} + \eta_i \eta_j [Y_i, Y_j] \right)_{G_+}$ denotes two elements in $G_+(A)$ as in (5.1).

This yields the functor $G_{\mathcal{P}}$ on objects, and one then defines it on morphisms in the obvious way. Namely, for any morphism $\varphi : A' \longrightarrow A''$ in $(\mathbf{Wsalg})_{\mathbb{K}}$ we let $G_{\mathcal{P}}(\varphi) : G_{\mathcal{P}}(A') \longrightarrow G_{\mathcal{P}}(A'')$ be the group morphism uniquely defined on generators by

$$\begin{aligned} G_{\mathcal{P}}(\varphi)(g'_+) &:= G_+(\varphi)(g'_+) & \forall \quad g'_+ \in G_+(A') \\ G_{\mathcal{P}}(\varphi)(1 + \eta' Y) &:= (1 + \varphi(\eta') Y) & \forall \quad \eta \in A'_1, Y \in \mathfrak{g}_1 \end{aligned}$$

As the defining relations of every group $G_{\mathcal{P}}(A)$ are independent of the chosen Weil superalgebra A , it follows that such a $G_{\mathcal{P}}(\varphi)$ is well defined indeed.

(b) We define a functor $G_{\mathcal{P}}^- : (\mathbf{Wsalg})_{\mathbb{K}} \longrightarrow (\mathbf{set})$ on any object $A \in (\mathbf{Wsalg})_{\mathbb{K}}$ by

$$G_{\mathcal{P}}^-(A) := \left\{ \prod_{s=1}^n (1 + \eta_s Y_s) \mid n \in \mathbb{N}, (\eta_s, Y_s) \in A_1 \times \mathfrak{g}_1 \quad \forall s \in \{1, \dots, n\} \right\} \quad (\subseteq G_{\mathcal{P}}(A))$$

and on morphism in the obvious way — just like for $G_{\mathcal{P}}$.

(c) Let us fix in \mathfrak{g}_1 a \mathbb{K} -basis $\{Y_i\}_{i \in I}$ — for some index set I — and a total order in I . We define a functor $G_{\mathcal{P}}^{\leq} : (\mathbf{Wsalg})_{\mathbb{K}} \longrightarrow (\mathbf{set})$ as follows. For $A \in (\mathbf{Wsalg})_{\mathbb{K}}$ we set

$$G_{\mathcal{P}}^{\leq}(A) := \left\{ \overrightarrow{\prod}_{i \in I} (1 + \eta_i Y_i) \mid \eta_i \in A_1 \quad \forall i \in I \right\} \quad (\subseteq G_{\mathcal{P}}^-(A) \subseteq G_{\mathcal{P}}(A))$$

where $\overrightarrow{\prod}_{i \in I}$ denotes an *ordered product* — with respect to the fixed total order in I . This defines the functor $G_{\mathcal{P}}^{\leq}$ on objects, and its definition on morphism is the obvious one (just like for $G_{\mathcal{P}}$). \diamond

Remarks 5.1.2.

(a) By their very definition, both $G_{\mathcal{P}}^-$ and $G_{\mathcal{P}}^{\leq}$ can be thought of as subfunctors of $G_{\mathcal{P}}$.

(b) It's easy to see that $G_{\mathcal{P}}^-(A)$ is the subgroup of $G_{\mathcal{P}}(A)$ generated by $\{(1 + \eta Y)\}_{(\eta, Y) \in A_1 \times \mathfrak{g}_1}$.

(c) By definition, the subfunctor $G_{\mathcal{P}}^{\leq}$ depends on the choice of the ordered basis $\{Y_i\}_{i \in I}$ of \mathfrak{g}_1 ; nevertheless, we shall presently see (in Proposition 5.2.1(a) later on) that this dependence is actually irrelevant for our purposes. Indeed, although $G_{\mathcal{P}}^{\leq}$ is definitely non-canonical, the supergroup subfunctor that it generates instead is *independent* of the choice of the ordered \mathbb{K} -basis of \mathfrak{g}_1 .

Next result shows that $G_{\mathcal{P}}$ can also be described using a much smaller set of generators:

Proposition 5.1.3. *Let $\mathcal{P} := (G_+, \mathfrak{g}) \in (\mathbf{sHCp})_{\mathbb{K}}$ be a super Harish-Chandra pair over \mathbb{K} ; also, we fix in \mathfrak{g}_1 a \mathbb{K} -basis $\{Y_i\}_{i \in I}$ — for some index set I — and a total order in I .*

Then for every Weil \mathbb{K} -superalgebra $A \in (\mathbf{Wsalg})_{\mathbb{K}}$ the group $G_{\mathcal{P}}(A)$ is generated by the set

$$\Gamma_A^{\diamond} := G_+(A) \cup \{(1 + \eta_i Y_i) \mid (\eta_i, Y_i) \in A_1 \times \mathfrak{g}_1, \forall i \in I\}$$

Proof. Given $A \in (\mathbf{Wsalg})_{\mathbb{K}}$, let $G_{\mathcal{P}}^{\diamond}(A)$ be the subgroup of $G_{\mathcal{P}}(A)$ generated by Γ_A^{\diamond} . We shall prove that every generator of the (larger, *a priori*) group $G_{\mathcal{P}}(A)$ of the form $(1 + \eta Y)$ with $(\eta, Y) \in A_1 \times \mathfrak{g}_1$ also belongs to the subgroup $G_{\mathcal{P}}^{\diamond}(A)$: this then will prove the claim.

So let $(\eta, Y) \in A_1 \times \mathfrak{g}_1$; then, in terms of the \mathbb{K} -basis $\{Y_i\}_{i \in I}$ of \mathfrak{g}_1 , our Y expands into $Y = \sum_{s=1}^k c_{j_s} Y_{j_s}$. By repeated applications of relations of the form $(1 + \eta Y') \cdot (1 + \eta Y'') = (1 + \eta(Y' + Y''))$, we find that the generator $(1 + \eta Y)$ in $G_{\mathcal{P}}(A)$ factors as

$$(1 + \eta Y) = \left(1 + \eta \sum_{s=1}^k c_{j_s} Y_{j_s}\right) = \prod_{s=1}^k (1 + c_{j_s} \eta Y_{j_s}) \quad (5.2)$$

where the product can be done in any order, as the factors in it mutually commute. Now the product in right-hand side does belong to $G_{\mathcal{P}}^{\diamond}(A)$, hence we are done. \square

5.1.4. Another realization of $G_{\mathcal{P}}$. Let $\mathcal{P} = (G_+, \mathfrak{g}) \in (\mathbf{sHCp})_{\mathbb{K}}$ be a super Harish-Chandra pair; we present now yet another way of realizing the \mathbb{K} -supergroup $G_{\mathcal{P}}$ introduced in Definition 5.1.1(a). In the following, if K is any group presented by generators and relations, we write $K = \langle \Gamma \rangle / (\mathcal{R})$ if Γ is a set of free generators (of K), \mathcal{R} is a set of “relations” among generators and (\mathcal{R}) is the normal subgroup in K generated by \mathcal{R} . As a matter of notation, given a presentation $K = \langle \Gamma \rangle / (\mathcal{R}) = \langle \Gamma \rangle / (\mathcal{R}_1 \cup \mathcal{R}_2)$ with $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$, the Double Quotient Theorem gives us

$$K = \langle \Gamma \rangle / (\mathcal{R}) = \langle \Gamma \rangle / (\mathcal{R}_1 \cup \mathcal{R}_2) = \langle \Gamma \rangle / (\mathcal{R}_1) / (\mathcal{R}_1 \cup \mathcal{R}_2) / (\mathcal{R}_1) = \langle \overline{\Gamma} \rangle / (\overline{\mathcal{R}_2}) \quad (5.3)$$

where $\overline{\Gamma}$ and $\overline{\mathcal{R}_2}$ respectively denote the images of Γ and of \mathcal{R}_2 in the quotient group $\langle \Gamma \rangle / (\mathcal{R}_1)$.

For any fixed $A \in (\mathbf{Wsalg})_{\mathbb{K}}$, we denote by $G_+^{[2]}(A)$ the subgroup of $G_+(A)$ generated by the set $\{(1 + cX) \mid c \in A_1^{[2]}, X \in [\mathfrak{g}_1, \mathfrak{g}_1]\}$ — cf. §2.1.1 for notation $A_1^{[2]}$. Note then that $G_+^{[2]}(A)$ is *normal* in $G_+(A)$, as one easily sees by construction (taking into account that, as $\mathcal{P} := (G_+, \mathfrak{g})$ is a super Harish-Chandra pair, the “adjoint” action of G_+ onto \mathfrak{g} maps $[\mathfrak{g}_1, \mathfrak{g}_1]$ into itself).

We consider also the three sets

$$\Gamma_A^+ := G_+(A), \quad \Gamma_A^{[2]} := G_+^{[2]}(A), \quad \Gamma_A^- := \Gamma_A^{[2]} \cup \{(1 + \eta Y)\}_{(\eta, Y) \in A_1 \times \mathfrak{g}_1}$$

and the sets of relations — for all $g_+, g'_+, g''_+ \in \Gamma_A^+$, $g_{[2]}, g'_{[2]}, g''_{[2]} \in \Gamma_A^{[2]}$, $\eta, \eta', \eta'' \in A_1$, $X \in [\mathfrak{g}_1, \mathfrak{g}_1]$, $Y, Y', Y'' \in \mathfrak{g}_1$, with $\dot{\cdot}_+$ and $\dot{\cdot}_{G_+^{[2]}}$ being the product in $G_+(A)$ and in $G_+^{[2]}(A)$ — given by

$$\mathcal{R}_A^+ : \quad g'_+ \cdot g''_+ = g'_+ \dot{\cdot}_+ g''_+$$

$$\mathcal{R}_A^- : \begin{cases} g'_{[2]} \cdot g''_{[2]} = g'_{[2]} \dot{g}_{+}^{[2]} g''_{[2]} \\ (1 + \eta Y) \cdot g_{[2]} = g_{[2]} \cdot (1 + \eta \text{Ad}(g_{[2]}^{-1})(Y)) \\ (1 + \eta'' Y) \cdot (1 + \eta' Y) = \left(1 + \eta' \eta'' Y^{\langle 2 \rangle}\right) \cdot (1 + (\eta' + \eta'') Y) \\ (1 + \eta'' Y'') \cdot (1 + \eta' Y') = \left(1 + \eta' \eta'' [Y', Y'']\right) \cdot (1 + \eta' Y') \cdot (1 + \eta'' Y'') \\ (1 + \eta Y') \cdot (1 + \eta Y'') = (1 + \eta (Y' + Y'')) \\ (1 + \eta 0_{\mathfrak{g}_1}) = 1 \quad , \quad (1 + 0_A Y) = 1 \end{cases}$$

$$\mathcal{R}_A^\times : \quad g_{[2]} \cdot g_+ = g_+ \cdot (g_+^{-1} \dot{g}_+ g_{[2]} \dot{g}_+ g_+) \quad , \quad (1 + \eta Y) \cdot g_+ = g_+ \cdot (1 + \eta \text{Ad}(g_+^{-1})(Y))$$

$$\mathcal{R}_A^{[2]} : \quad (g_{[2]})_{\Gamma_A^{[2]}} = (g_{[2]})_{\Gamma_A^+}$$

$$\mathcal{R}_A := \mathcal{R}_A^+ \cup \mathcal{R}_A^- \cup \mathcal{R}_A^\times \cup \mathcal{R}_A^{[2]}$$

(in particular, note that the relations of type $\mathcal{R}_A^{[2]}$ in down-to-earth terms just identify each element in $\Gamma_A^{[2]}$ with its corresponding copy inside Γ_A^+). Then we *define* a new group, by generators and relations, namely $G_p^-(A) := \langle \Gamma_A^- \rangle / (\mathcal{R}_A^-)$.

From the very definition of $G_p(A)$ — cf. Definition 5.1.1 — it follows that

$$G_p(A) \cong \langle \Gamma_A^+ \cup \Gamma_A^- \rangle / (\mathcal{R}_A) = \langle \Gamma_A^+ \cup \Gamma_A^- \rangle / \left(\mathcal{R}_A^+ \cup \mathcal{R}_A^- \cup \mathcal{R}_A^\times \cup \mathcal{R}_A^{[2]} \right) \quad (5.4)$$

indeed, here above we are just taking larger sets of generators and of relations (w.r.t. Definition 5.1.1), but with enough redundancies as to find a different presentation of *the same* group.

From this we find a neat description of $G_p(A)$ by achieving the presentation (5.4) in a series of intermediate steps, namely adding only one bunch of relations at a time. As a first step, we have

$$\langle \Gamma_A^+ \cup \Gamma_A^- \rangle / (\mathcal{R}_A^+ \cup \mathcal{R}_A^-) = \langle \Gamma_A^+ \rangle / (\mathcal{R}_A^+) * \langle \Gamma_A^- \rangle / (\mathcal{R}_A^-) \cong G_+(A) * G_p^-(A) \quad (5.5)$$

where $G_+(A) \cong \langle \Gamma_A^+ \rangle / (\mathcal{R}_A^+)$ by construction and $*$ denotes the free product (of two groups).

For the next two steps we can follow two different lines of action. On the one hand, one has

$$\langle \Gamma_A^+ \cup \Gamma_A^- \rangle / (\mathcal{R}_A^+ \cup \mathcal{R}_A^- \cup \mathcal{R}_A^\times) \cong \left(G_+(A) * G_p^-(A) \right) / \left(\overline{\mathcal{R}_A^\times} \right) \cong G_+(A) \ltimes G_p^-(A)$$

because of (5.3) and (5.5) together, where $G_+(A) \ltimes G_p^-(A)$ is the semidirect product of $G_+(A)$ with $G_p^-(A)$ with respect to the obvious (“adjoint”) action of the former on the latter. Then

$$\begin{aligned} \langle \Gamma_A^+ \cup \Gamma_A^- \rangle / (\mathcal{R}_A) &\cong \langle \Gamma_A^+ \cup \Gamma_A^- \rangle / \left(\mathcal{R}_A^+ \cup \mathcal{R}_A^- \cup \mathcal{R}_A^\times \cup \mathcal{R}_A^{[2]} \right) \cong \\ &\cong \left(G_+(A) \ltimes G_p^-(A) \right) / \left(\overline{\mathcal{R}_A^{[2]}} \right) \cong \left(G_+(A) \ltimes G_p^-(A) \right) / N_{[2]}(A) \end{aligned}$$

where $N_{[2]}(A)$ is the normal subgroup of $G_+(A) \ltimes G_p^-(A)$ generated by $\left\{ (g_{[2]}, g_{[2]}^{-1}) \right\}_{g_{[2]} \in \Gamma_A^{[2]}}$.

This together with (5.4) eventually yields

$$G_p(A) = \left(G_+(A) \ltimes G_-(A) \right) / N_{[2]}(A)$$

On the other hand, again from (5.3) and (5.5) together we get

$$\langle \Gamma_A^+ \cup \Gamma_A^- \rangle / \left(\mathcal{R}_A^+ \cup \mathcal{R}_A^- \cup \mathcal{R}_A^{[2]} \right) \cong \left(G_+(A) * G_{\mathcal{P}}^-(A) \right) / \left(\overline{\mathcal{R}_A^{[2]}} \right) \cong G_+(A) *_{G_+^{[2]}(A)} G_{\mathcal{P}}^-(A)$$

where $G_+(A) *_{G_+^{[2]}(A)} G_{\mathcal{P}}^-(A)$ is the amalgamated product of $G_+(A)$ and $G_{\mathcal{P}}^-(A)$ over $G_+^{[2]}(A)$ w.r.t. the obvious natural monomorphisms $G_+^{[2]}(A) \hookrightarrow G_+(A)$ and $G_+^{[2]}(A) \hookrightarrow G_{\mathcal{P}}^-(A)$. Then

$$\begin{aligned} \langle \Gamma_A^+ \cup \Gamma_A^- \rangle / (\mathcal{R}_A) &\cong \langle \Gamma_A^+ \cup \Gamma_A^- \rangle / \left(\mathcal{R}_A^+ \cup \mathcal{R}_A^- \cup \mathcal{R}_A^{[2]} \cup \mathcal{R}_A^{\times} \right) \cong \\ &\cong \left(G_+(A) *_{G_+^{[2]}(A)} G_{\mathcal{P}}^-(A) \right) / \left(\overline{\mathcal{R}_A^{\times}} \right) \cong \left(G_+(A) *_{G_+^{[2]}(A)} G_{\mathcal{P}}^-(A) \right) / N_{\times}(A) \end{aligned}$$

where $N_{\times}(A)$ is the normal subgroup of $G_+(A) *_{G_+^{[2]}(A)} G_{\mathcal{P}}^-(A)$ generated by

$$\begin{aligned} &\left\{ g_+ \cdot (1 + \eta Y) \cdot g_+^{-1} \cdot (1 + \eta \text{Ad}(g_+)(Y))^{-1} \right\}_{(\eta, Y) \in A_1 \times \mathfrak{g}_1, g_+ \in G_+(A)} \cup \\ &\cup \left\{ g_+ \cdot g_{[2]} \cdot g_+ \cdot (g_+ \dot{g}_+ g_{[2]} \dot{g}_+ g_+)^{-1} \right\}_{g_+ \in G_+(A), g_{[2]} \in \Gamma_A^{[2]}} \end{aligned}$$

All this along with (5.4) eventually gives

$$G_{\mathcal{P}}(A) = \left(G_+(A) *_{G_+^{[2]}(A)} G_{\mathcal{P}}^-(A) \right) / N_{\times}(A)$$

for all $A \in (\mathbf{Wsalg})_{\mathbb{K}}$. In functorial terms this yields

$$\begin{aligned} G_{\mathcal{P}} &= \left(G_+ \ltimes G_{\mathcal{P}}^- \right) / N_{[2]} \quad \text{and} \quad G_{\mathcal{P}} = \left(G_+ *_{G_+^{[2]}} G_{\mathcal{P}}^- \right) / N_{\times} \\ \text{or} \quad G_{\mathcal{P}} &= G_+ \ltimes_{G_+^{[2]}} G_{\mathcal{P}}^- \end{aligned}$$

where the last, (hopefully) more suggestive notation $G_{\mathcal{P}} = G_+ \ltimes_{G_+^{[2]}} G_{\mathcal{P}}^-$ tells us that $G_{\mathcal{P}}$ is the “amalgamate semidirect product” of G_+ and $G_{\mathcal{P}}^-$ over their common subgroup $G_+^{[2]}$.

The above construction(s) of $G_{\mathcal{P}}$ provides us with a supergroup functor, for each super Harish-Chandra pair \mathcal{P} : next subsection is devoted to prove that this is indeed a *Lie supergroup*.

5.2 The supergroup functor $G_{\mathcal{P}}$ as a Lie supergroup

We aim now to proving that the supergroup functor $G_{\mathcal{P}}$ is actually a Lie supergroup; for this, we need to investigate its structure in some detail. We keep definitions and notations as given before: in particular, recall that for any $A \in (\mathbf{Wsalg})_{\mathbb{K}}$ we denote by $G_+^{[2]}(A)$ the subgroup of $G_+(A)$ generated by $\{ (1 + cX) \mid c \in A_1^{[2]}, X \in [\mathfrak{g}_1, \mathfrak{g}_1] \}$ — cf. §2.1.1 for notation $A_1^{[2]}$.

Our first step is the following “factorization result” for $G_{\mathcal{P}}$:

Proposition 5.2.1. *Let $\mathcal{P} := (G_+, \mathfrak{g}) \in (\mathbf{sHCp})_{\mathbb{K}}$ be a super Harish-Chandra pair over \mathbb{K} , let $\{Y_i\}_{i \in I}$ be a totally ordered \mathbb{K} -basis of \mathfrak{g}_1 (for our fixed order in I) and $A \in (\mathbf{Wsalg})_{\mathbb{K}}$. Then:*

(a) *letting $\langle G_{\leq}(A) \rangle$ be the subgroup of $G_{\mathcal{P}}(A)$ generated by $G_{\leq}(A)$, we have*

$$\langle G_{\leq}(A) \rangle = G_{\mathcal{P}}^-(A)$$

and there exist group-theoretic factorizations

$$G_{\mathcal{P}}^-(A) = G_+^{[2]}(A) \cdot G_{\leq}(A) \quad , \quad G_{\mathcal{P}}^-(A) = G_{\leq}(A) \cdot G_+^{[2]}(A)$$

(b) *there exist group-theoretic factorizations*

$$G_{\mathcal{P}}(A) = G_+(A) \cdot G_{\leq}(A) \quad , \quad G_{\mathcal{P}}(A) = G_{\leq}(A) \cdot G_+(A)$$

Proof. Claim (a) is the exact analogue of Proposition 4.2.6(a), and (b) the analogue of (4.12) in Proposition 4.2.6(b). In both cases the proof (up to trivialities) is identical, so we can skip it. \square

5.2.2. The representation $G_{\mathcal{P}} \longrightarrow \mathrm{GL}(V)$. When discussing the structure of a Lie supergroup G , the factorization $G = G_0 \cdot G_{\leq}$ was just a intermediate step; Proposition 5.2.1 above gives us the parallel counterpart for $G_{\mathcal{P}}$. Such a result for G is improved into the “Global Splitting Theorem” — i.e. Theorem 4.2.9 — that, roughly speaking, states that for any $g \in G(A)$ the factorization pertaining to $G_0(A) \cdot G_{\leq}(A)$ has uniquely determined factors, and similarly any element in $G_{\leq}(A)$ has a unique factorization into an ordered product of factors of the form $(1 + \eta_i Y_i)$. Both results are proved by showing that two factorizations of the same object necessarily have identical factors; in other words, *distinct* choices of factors always give rise to *different* elements in $G(A)$ or in $G_{\leq}(A)$. This last fact was proved using the concrete realization of $G(A)$ as a special set of maps, namely $G(A) := \coprod_{x \in |G|} \mathrm{Hom}_{(\mathbf{salg})_{\mathbb{K}}}(\mathcal{O}_{|G|,x}, A)$; indeed, this algebra is rich enough to “separate” different elements of $G(A)$ itself just looking at their values as A -valued maps. When dealing with $G_{\mathcal{P}}(A)$ instead, that is defined abstractly (by generators and relations), such a built-in realization is not available from scratch: our strategy then is to replace it with a suitable “partial linearization” of $G_{\mathcal{P}}(A)$, namely with a representation of that group that, though not being faithful, is still “rich enough” to (almost) separate elements so that uniqueness of factorizations can be proved again.

Let $\mathcal{P} = (G_+, \mathfrak{g}) \in (\mathbf{sHCp})_{\mathbb{K}}$ be our given super Harish-Chandra pair over \mathbb{K} ; as before, we fix a \mathbb{K} -basis $\{Y_i\}_{i \in I}$ of \mathfrak{g} , where I is an index set in which we fix some total order, hence the basis itself is totally ordered as well.

Recall that the *universal enveloping algebra* $U(\mathfrak{g})$ is given by $U(\mathfrak{g}) := T(\mathfrak{g})/J$ where $T(\mathfrak{g})$ is the tensor algebra of \mathfrak{g} and J is the two-sided ideal in $T(\mathfrak{g})$ generated by the set

$$\left\{ xy - (-1)^{|x||y|} yx - [x, y], z^2 - z^{\langle 2 \rangle} \mid x, y \in \mathfrak{g}_0 \cup \mathfrak{g}_1, z \in \mathfrak{g}_1 \right\}$$

— where $z^{\langle 2 \rangle} := 2^{-1}[z, z]$, see Definition 2.2.1(c). It is known then — see for instance [25], §7.2, which clearly adapt to the complex case too — that one has splitting(s) of \mathbb{K} -supercoalgebras

$$U(\mathfrak{g}) \cong U(\mathfrak{g}_0) \otimes_{\mathbb{K}} \bigwedge \mathfrak{g}_1 \cong \bigwedge \mathfrak{g}_1 \otimes_{\mathbb{K}} U(\mathfrak{g}_0) \quad (5.6)$$

In addition, $\bigwedge \mathfrak{g}_1$ has \mathbb{K} -basis $\{Y_{i_1} Y_{i_2} \cdots Y_{i_s} \mid s \leq |I|, i_1 < i_2 < \cdots < i_s\}$; hereafter, we drop the sign “ \wedge ” to denote the product in $\bigwedge \mathfrak{g}_1$.

Now let $\mathbf{1}$ be the (one-dimensional) *trivial representation* of \mathfrak{g}_0 . By the standard process of *induction* from \mathfrak{g}_0 to \mathfrak{g} — the former being thought of as a Lie subsuperalgebra of the latter — we

can consider the *induced representation* $V := \text{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}}(\mathbf{1})$, that is a \mathfrak{g} -module. Looking at $\mathbf{1}$ and V respectively as a module over $U(\mathfrak{g}_0)$ and over $U(\mathfrak{g})$, taking (5.6) into account we get

$$V := \text{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}}(\mathbf{1}) = U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_0)} \mathbf{1} \cong \bigwedge_{\mathbb{K}} \mathfrak{g}_1 \otimes_{\mathbb{K}} \mathbf{1} \cong \bigwedge_{\mathbb{K}} \mathfrak{g}_1 \quad (5.7)$$

The last one above is a natural isomorphism of \mathbb{K} -superspaces, uniquely determined once a specific element $\underline{b} \in \mathbf{1}$ is fixed to form a \mathbb{K} -basis of $\mathbf{1}$ itself: the isomorphism is $\omega \otimes \underline{b} \mapsto \omega$ for all $\omega \in \bigwedge_{\mathbb{K}} \mathfrak{g}_1$.

This representation-theoretical construction and its outcome clearly give rise to similar functorial counterparts, for the Lie algebra valued \mathbb{K} -superfunctors $\mathcal{L}_{\mathfrak{g}_0}$ and $\mathcal{L}_{\mathfrak{g}}$, as well as for the \mathbb{K} -superfunctors associated with $U(\mathfrak{g}_0)$ and $U(\mathfrak{g})$, in the standard way, namely $A \mapsto A_0 \otimes_{\mathbb{K}} U(\mathfrak{g}_0)$ and $A \mapsto (A \otimes_{\mathbb{K}} U(\mathfrak{g}))_0 = A_0 \otimes_{\mathbb{K}} U(\mathfrak{g})_0 \oplus A_1 \otimes_{\mathbb{K}} U(\mathfrak{g})_1$ for all $A \in (\mathbf{Wsalg})_{\mathbb{K}}$.

On the other hand, recall that $\mathfrak{g}_0 = \text{Lie}(G_+)$, and clearly $\mathbf{1}$ is also the trivial representation for G_+ , as a classical Lie group — of real smooth, real analytic or complex holomorphic type. Then, by construction and by (4.8), it is clear that the representation of \mathfrak{g} on the space V also induce a representation of the super Harish-Chandra pair $\mathcal{P} = (G_+, \mathfrak{g})$ on the same V , in other words V bears also a structure of (G_+, \mathfrak{g}) -module — in the obvious, natural sense: we have a morphism $(\mathbf{r}_+, \rho) : (G_+, \mathfrak{g}) \longrightarrow (\text{GL}(V), \mathfrak{gl}(V))$ of super Harish-Chandra pairs. We shall also use again ρ to denote the representation map $\rho : U(\mathfrak{g}) \longrightarrow \text{End}_{\mathbb{K}}(V)$ giving the $U(\mathfrak{g})$ -module structure on V .

Our key step now is to remark that the above (G_+, \mathfrak{g}) -module structure on V actually “integrates” to a $G_{\mathcal{P}}$ -module structure, in a natural way.

Proposition 5.2.3. *Retain notation as above for the (G_+, \mathfrak{g}) -module V . There exists a unique structure of (left) $G_{\mathcal{P}}$ -module onto V which satisfies the following conditions: for every $A \in (\mathbf{Wsalg})_{\mathbb{K}}$, the representation map $\mathbf{r}_{\mathcal{P},A} : G_{\mathcal{P}}(A) \longrightarrow \text{GL}(V)(A)$ is given on generators of $G_{\mathcal{P}}(A)$ — namely, all $g_+ \in G_+(A)$ and $(1 + \eta_i Y_i)$ for $i \in I$, $\eta_i \in A_1$ — by*

$$\mathbf{r}_{\mathcal{P},A}(g_+) := \mathbf{r}_+(g_+) \quad , \quad \mathbf{r}_{\mathcal{P},A}(1 + \eta_i Y_i) := \rho(1 + \eta_i Y_i) = \text{id}_V + \eta_i \rho(Y_i)$$

or, in other words, $g_+.v := \mathbf{r}_+(g_+)(v)$ and $(1 + \eta_i Y_i).v := \rho(1 + \eta_i Y_i)(v) = v + \eta_i \rho(Y_i)(v)$ for all $v \in V(A)$. Overall, this yields a morphism a \mathbb{K} -supergroup functors $\mathbf{r}_{\mathcal{P}} : G_{\mathcal{P}} \longrightarrow \text{GL}(V)$.

Proof. This is, essentially, a straightforward consequence of the whole construction, and of the very definition of $G_{\mathcal{P}}$. Indeed, by definition of representation for the super Harish-Chandra pair \mathcal{P} we see that the operators $\mathbf{r}_{\mathcal{P},A}(g_+)$ and $\mathbf{r}_{\mathcal{P},A}(1 + \eta_i Y_i)$ on V — associated with the generators of $G_{\mathcal{P}}(A)$ — do satisfy all relations which, by Definition 5.1.1(a), are satisfied by the generators themselves. Thus they uniquely provide a well-defined group morphism $\mathbf{r}_{\mathcal{P},A} : G_{\mathcal{P}}(A) \longrightarrow \text{GL}(V)(A)$ as required. The construction is clearly functorial in A , whence the claim. \square

The advantage of introducing the representation $\mathbf{r}_{\mathcal{P}}$ of $G_{\mathcal{P}}$ on V is that it allows us to “separate”, in a sense, the “odd points of $G_{\mathcal{P}}(A)$ from each other and from the even ones”, i.e. we can separate the points in $G_{\mathcal{P}}^{\leq}(A)$ from each other (in a “very fine” sense) and from those in $G_+(A)$. Before seeing that, we need yet an additional, technical result, the counterpart for $G_{\mathcal{P}}(A)$ of Lemma 4.2.7:

Lemma 5.2.4. *Let $A \in (\mathbf{Wsalg})_{\mathbb{K}}$, let $\hat{\eta}_i, \check{\eta}_i \in A_1$ and let \mathfrak{a} be an ideal of A such that $\hat{\eta}_i, \check{\eta}_i \in \mathfrak{a}$ and $\alpha_i := \hat{\eta}_i - \check{\eta}_i \in \mathfrak{a}^n$ ($i \in I$) for some $n \in \mathbb{N}_+$. Then in the group $G_{\mathcal{P}}(A/\mathfrak{a}^{n+1})$ we have*

$$\overrightarrow{\prod}_{i \in I} (1 + [\hat{\eta}_i]_{n+1} Y_i) \cdot \overleftarrow{\prod}_{i \in I} (1 - [\check{\eta}_i]_{n+1} Y_i) = \overrightarrow{\prod}_{i \in I} (1 + [\alpha_i]_{n+1} Y_i) \in G_{\mathcal{P}}^{\leq}(A/\mathfrak{a}^{n+1})$$

where $\overrightarrow{\prod}_{i \in I}$ and $\overleftarrow{\prod}_{i \in I}$ respectively denote an ordered and a reversely-ordered product (w.r. to the given order in I) and $[a]_{n+1} \in A/\mathfrak{a}^{n+1}$ stands for the coset modulo \mathfrak{a}^{n+1} of any $a \in A$.

Proof. Just like for Lemma 4.2.7, this is a direct consequence of the defining relations of $G_p(A)$ — which fix the “commutation rules” among generators of type $(1 + \eta Y)$ and/or $(1 + a Y^{(2)})_{G_+}$ and/or $(1 + c[Y', Y''])_{G_+}$ — and of the fact that the assumptions imply, for all $i \in I$,

$$\hat{\eta}_i \check{\eta}_i = \hat{\eta}_i (\hat{\eta}_i - \alpha_i) = \hat{\eta}_i^2 - \hat{\eta}_i \alpha_i = -\hat{\eta}_i \alpha_i \in \mathfrak{a}^{n+1} . \quad \square$$

We are now ready to state and prove the main result of the present subsection, that is just the “global splitting theorem” for G_p (cf. Theorem 4.2.9):

Proposition 5.2.5.

(a) *The restriction of group multiplication in G_p provides isomorphisms of (set-valued) functors*

$$G_+ \times G_- \cong G_p , \quad G_- \times G_+ \cong G_p , \quad G_+^{[2]} \times G_- \cong G_p^- , \quad G_- \times G_+^{[2]} \cong G_p^-$$

(b) *There exists an isomorphism of (set-valued) functors $\mathbb{A}_{\mathbb{K}}^{0|d_-} \cong G_-$, with $d_- := |I| = \dim_{\mathbb{K}}(\mathfrak{g}_1)$, given on A -points — for every $A \in (\mathbf{Wsalg})_{\mathbb{K}}$ — by*

$$\mathbb{A}_{\mathbb{K}}^{0|d_-}(A) = A_1^{d_-} \longrightarrow G_-(A) , \quad (\eta_i)_{i \in I} \mapsto \prod_{i \in I} (1 + \eta_i Y_i)$$

(c) *There exist isomorphisms of (set-valued) functors*

$$G_+ \times \mathbb{A}_{\mathbb{K}}^{0|d_-} \cong G_p , \quad G_+^{[2]} \times \mathbb{A}_{\mathbb{K}}^{0|d_-} \cong G_p^- , \quad \text{and} \quad \mathbb{A}_{\mathbb{K}}^{0|d_-} \times G_+ \cong G_p , \quad \mathbb{A}_{\mathbb{K}}^{0|d_-} \times G_+^{[2]} \cong G_p^-$$

given on A -points — for every $A \in (\mathbf{Wsalg})_{\mathbb{K}}$ — respectively by

$$(g_+ , (\eta_i)_{i \in I}) \mapsto g_+ \cdot \prod_{i \in I} (1 + \eta_i Y_i) \quad \text{and} \quad ((\eta_i)_{i \in I} , g_+) \mapsto \prod_{i \in I} (1 + \eta_i Y_i) \cdot g_+$$

Proof. The proof is quite close to that of Theorem 4.2.9, but with some technical differences, involving the use of the representation V of §5.2.2; for completeness we present it explicitly.

(a) It is enough to prove the first identity concerning G_p , as all other are similar. Thus our goal amounts to showing the following: for any $A \in (\mathbf{Wsalg})_{\mathbb{K}}$, if $\hat{g}_+ \hat{g}_- = \check{g}_+ \check{g}_-$ for $\hat{g}_+ , \check{g}_+ \in G_+(A)$ and $\hat{g}_- , \check{g}_- \in G_-(A)$, then $\hat{g}_+ = \check{g}_+$ and $\hat{g}_- = \check{g}_-$.

The assumption $\hat{g}_+ \hat{g}_- = \check{g}_+ \check{g}_-$ implies $g := \hat{g}_- \check{g}_-^{-1} = \hat{g}_+^{-1} \check{g}_+ \in G_+(A)$, as $G_+(A)$ is a subgroup in $G_p(A)$. Now $\hat{g}_- \in G_-(A)$ has the form $\hat{g}_- = \prod_{i \in I} (1 + \hat{\eta}_i Y_i)$ and similarly $\check{g}_- = \prod_{i \in I} (1 + \check{\eta}_i Y_i)$ so that $\check{g}_-^{-1} = \prod_{i \in I} (1 - \check{\eta}_i Y_i)$; therefore we have

$$g := \hat{g}_- \check{g}_-^{-1} = \prod_{i \in I} (1 + \hat{\eta}_i Y_i) \prod_{i \in I} (1 - \check{\eta}_i Y_i) \in G_+(A) \subseteq G_p(A) \quad (5.8)$$

Let $\mathfrak{a} := (\{\hat{\eta}_i, \check{\eta}_i\}_{i \in I})$ be the ideal of A generated by the $\hat{\eta}_i$ ’s and the $\check{\eta}_i$ ’s, set $A \xrightarrow{\pi_n} A/\mathfrak{a}^n$ for the quotient map and $[a]_n := \pi_n(a)$ for $a \in A$, then $G_p(A) \xrightarrow{G(\pi_n)} G_p(A/\mathfrak{a}^n)$ for the associated group morphism and $[y]_n := G_p(\pi_n)(y)$ for every $y \in G_p(A)$. Now (5.8) along with Lemma 5.2.4 for $n := 1$, letting $\alpha_i := \hat{\eta}_i - \check{\eta}_i \in \mathfrak{a}$ for all $i \in I$, gives

$$[g]_2 = \prod_{i \in I} (1 + [\hat{\eta}_i]_2 Y_i) \cdot \prod_{i \in I} (1 - [\check{\eta}_i]_2 Y_i) = \prod_{i \in I} (1 + [\alpha_i]_2 Y_i) \in G_-(A/\mathfrak{a}^2) \quad (5.9)$$

Next step then is to let $[g]_2$ act onto $\underline{b} \in V(A/\mathfrak{a}^2)$. In order to avoid confusion, when we describe V as $V = \bigwedge \mathfrak{g}_1 \cdot \underline{b} \cong \bigwedge \mathfrak{g}_1$, we write the elements of the \mathbb{K} -basis $\{Y_i\}_{i \in I}$ of \mathfrak{g}_1 as \bar{Y}_i instead of Y_i : thus the \mathbb{K} -linear isomorphism $\bigwedge \mathfrak{g}_1 \cdot \underline{b} \cong \bigwedge \mathfrak{g}_1$ is given by $(Y_{i_1} Y_{i_2} \cdots Y_{i_s}) \cdot \underline{b} \mapsto \bar{Y}_{i_1} \bar{Y}_{i_2} \cdots \bar{Y}_{i_s}$ — for all $i_1 < i_2 < \cdots < i_s$.

Taking into account that $[\alpha_h]_2[\alpha_k]_2 = [0]_2 \in A/\mathfrak{a}^2$ (for all $h, k \in I$) from (5.9) we get that the action of $[g]_2$ onto $\underline{b} \in V(A/\mathfrak{a}^2)$ is given by

$$[g]_2 \cdot \underline{b} = \prod_{i \in I}^{\rightarrow} (1 + [\alpha_i]_2 Y_i) \cdot \underline{b} = 1 \cdot \underline{b} + \sum_{i \in I} [\alpha_i]_2 Y_i \cdot \underline{b} = \underline{b} + \sum_{i \in I} [\alpha_i]_2 \bar{Y}_i \in V(A/\mathfrak{a}^2) \quad (5.10)$$

On the other hand, we have also $[g]_2 \cdot \underline{b} = \underline{b}$ because $[g]_2 \in G_+(A/\mathfrak{a}^2)$ and G_+ acts trivially on V . This compared with (5.10), taking into account that $\{\underline{b}\} \cup \{\bar{Y}_i\}_{i \in I}$ is part of the chosen basis of V , implies that $[\alpha_i]_2 = [0]_2 \in A/\mathfrak{a}^2$, i.e. $\alpha_i \in \mathfrak{a}^2$, for all $i \in I$. But then we can repeat the previous argument: indeed, we can now apply Lemma 5.2.4 again with $n := 2$, and get the analogue of (5.9), where $2 = 1 + 1$ is replaced everywhere by $3 = 2 + 1 = n + 1$, namely

$$[g]_3 = \prod_{i \in I}^{\rightarrow} (1 + [\hat{\eta}_i]_3 Y_i) \prod_{i \in I}^{\leftarrow} (1 - [\check{\eta}_i]_3 Y_i) = \prod_{i \in I}^{\rightarrow} (1 + [\alpha_i]_3 Y_i) \in G_{\mathcal{P}}(A/\mathfrak{a}^3) \quad (5.11)$$

Then we repeat the second step, namely we let $[g]_3$ act onto $\underline{b} \in V(A/\mathfrak{a}^3)$, for which (5.11) gives the analogue of (5.10), namely

$$[g]_3 \cdot \underline{b} = \underline{b} + \sum_{i \in I} [\alpha_i]_3 \bar{Y}_i \in V(A/\mathfrak{a}^3)$$

which in turn implies $\alpha_i \in \mathfrak{a}^3$, for all $i \in I$. Clearly, we can iterate this process, and find $\alpha_i \in \mathfrak{a}^n$ for all $n \in \mathbb{N}$, $i \in I$; as $\mathfrak{a}^n = \{0\}$ for $n \gg 0$ (since \mathfrak{a} is generated by finitely many odd elements) we end up with $\alpha_i = 0$, i.e. $\hat{\eta}_i = \check{\eta}_i$, for all $i \in I$. This means $\hat{g}_- = \check{g}_-$, whence $\hat{g}_+ = \check{g}_+$ as well.

(b) By construction there exists a morphism $\Theta : \mathbb{A}_{\mathbb{K}}^{0|d-} \longrightarrow G_{\leq}^{\leq}$ of set-valued functors that is given on A -points — for every single $A \in (\mathbf{Wsalg})_{\mathbb{K}}$ — by the map

$$\Theta_A : \mathbb{A}_{\mathbb{K}}^{0|d-}(A) := A_1^{\times d-} \longrightarrow G_{\leq}^{\leq}(A) , \quad (\eta_i)_{i \in I} \mapsto \Theta_A((\eta_i)_{i \in I}) := \prod_{i \in I}^{\rightarrow} (1 + \eta_i Y_i)$$

that is actually *surjective*. We need to prove that all these maps Θ_A are also injective, so that overall Θ is indeed an isomorphism.

Let $(\hat{\eta}_i)_{i \in I}, (\check{\eta}_i)_{i \in I} \in A_1^{\times d-}$ be such that $\Theta_A((\hat{\eta}_i)_{i \in I}) = \Theta_A((\check{\eta}_i)_{i \in I})$, that is $\prod_{i \in I}^{\rightarrow} (1 + \hat{\eta}_i Y_i) = \prod_{i \in I}^{\rightarrow} (1 + \check{\eta}_i Y_i)$. Then we can replay the proof of claim (a), now with $\hat{g}_+ := 1 =: \check{g}_+$: the outcome will be again $\hat{\eta}_i = \check{\eta}_i$ for all $i \in I$, i.e. $(\hat{\eta}_i)_{i \in I} = (\check{\eta}_i)_{i \in I}$. Thus Θ_A is injective, as desired.

(c) This is a direct consequence of (a) and (b) together. \square

5.2.6. The Lie supergroup structure of $G_{\mathcal{P}}$. For any given $\mathcal{P} \in (\mathbf{sHCp})_{\mathbb{K}}$ and $A \in (\mathbf{Wsalg})_{\mathbb{K}}$, consider the group $G_{\mathcal{P}}(A)$. Thanks to Proposition 5.2.5(c), we have a particular bijection

$$\phi_A : G_+(A) \times \mathbb{A}_{\mathbb{K}}^{0|d-}(A) \xrightarrow{\cong} G_{\mathcal{P}}(A) \quad (5.12)$$

whose restriction to $G_+(A)$, identified with $G_+(A) \times \{(0)_{i \in I}\} \subseteq G_+(A) \times \mathbb{A}_{\mathbb{K}}^{0|d-}(A)$, is the identity — onto the copy of $G_+(A)$ naturally sitting inside $G_{\mathcal{P}}(A)$.

Now, $G_+(A)$ is by definition an A_0 -manifold (cf. §2.4.4), of the same type (real smooth, etc.) as the super Harish-Chandra pair $\mathcal{P} := (G_+, \mathfrak{g})$ it pertains to; on the other hand, $\mathbb{A}_{\mathbb{K}}^{0|d-}(A)$ carries natural, canonical structures of A_0 -manifold of any possible type (real smooth or real analytic if $\mathbb{K} = \mathbb{R}$, complex holomorphic if $\mathbb{K} = \mathbb{C}$), in particular then also of the type of $G_+(A)$. Then we know that there is also a canonical “product structure” of A_0 -manifold — of the same type

of $G_+(A)$, i.e. of \mathcal{P} — onto the Cartesian product $G_+(A) \times \mathbb{A}_{\mathbb{K}}^{0|d-}(A)$. Using the bijection ϕ_A in (5.12) we push-forward this canonical \mathcal{A}_0 -manifold structure of $G_+(A) \times \mathbb{A}_{\mathbb{K}}^{0|d-}(A)$ onto $G_{\mathcal{P}}(A)$, which then is turned into an \mathcal{A}_0 -manifold on its own, still of the same type as \mathcal{P} .

Strictly speaking, the structure of \mathcal{A}_0 -manifold defined on $G_{\mathcal{P}}(A)$ formally depends on the choice of G_{\leq} , hence of a totally ordered \mathbb{K} -basis of \mathfrak{g}_1 , as this choice enters in the construction of ϕ_A in (5.12) above. However, thanks to the special form of the defining relations of $G_{\mathcal{P}}(A)$ it is straightforward to show that *changing such a basis amounts to changing local charts for the same, unique \mathcal{A}_0 -manifold structure*; so in the end the structure is actually independent of such a choice.

Now, using the above mentioned structure of \mathcal{A}_0 -manifold on $G_{\mathcal{P}}(A)$ for each $A \in (\mathbf{Wsalg})_{\mathbb{K}}$, given a morphism $f : A' \rightarrow A''$ in $(\mathbf{Wsalg})_{\mathbb{K}}$ it is straightforward to check that the corresponding group morphism $G_{\mathcal{P}}(f) : G_{\mathcal{P}}(A') \rightarrow G_{\mathcal{P}}(A'')$ is a morphism of \mathcal{A}_0 -manifolds, hence it is a morphism of \mathcal{A}_0 -manifolds (cf. §2.4.4). Thus $G_{\mathcal{P}}$ can also be seen as a functor from Weil \mathbb{K} -superalgebras to \mathcal{A}_0 -manifolds (real smooth, real analytic or complex holomorphic as \mathcal{P} is).

At last, again looking at the commutation relations in $G_{\mathcal{P}}(A)$, we see that the group multiplication and the inverse map are “regular” (that is to say, “real smooth”, “real analytic” or “complex holomorphic” depending on the type of \mathcal{P}); indeed, this is explicitly proved by calculations like those needed in the proof of Proposition 5.2.1(b) — that we skipped, so refer instead to the proof of Proposition 4.2.6(b). Thus they are morphisms of \mathcal{A}_0 -manifolds, so $G_{\mathcal{P}}(A)$ is a group element among \mathcal{A}_0 -manifolds, i.e. it is a Lie \mathcal{A}_0 -group; hence (cf. Proposition 2.4.7), overall the functor $G_{\mathcal{P}}$ is a Lie supergroup, of real smooth, real analytic or complex holomorphic type as \mathcal{P} is.

Eventually, the outcome of this discussion — and core result of the present section — is the following statement, which provides a “backward functor” from sHCp’s to Lie supergroups:

Theorem 5.2.7. *The recipe in Definition 5.1.1 provides functors*

$$\Psi : (\mathbf{sHCp})_{\mathbb{R}}^{\infty} \rightarrow (\mathbf{Lsgrp})_{\mathbb{R}}^{\infty} \quad , \quad \Psi : (\mathbf{sHCp})_{\mathbb{R}}^{\omega} \rightarrow (\mathbf{Lsgrp})_{\mathbb{R}}^{\omega} \quad , \quad \Psi : (\mathbf{sHCp})_{\mathbb{C}}^{\omega} \rightarrow (\mathbf{Lsgrp})_{\mathbb{C}}^{\omega}$$

given on objects by $\mathcal{P} \mapsto \Psi(\mathcal{P}) := G_{\mathcal{P}}$ and on morphisms by

$$\left((\Omega_+, \omega) : \mathcal{P}' \rightarrow \mathcal{P}'' \right) \mapsto \left(\Psi((\Omega_+, \omega)) : \Psi(\mathcal{P}') := G_{\mathcal{P}'} \rightarrow G_{\mathcal{P}''} =: \Psi(\mathcal{P}'') \right)$$

where the functor morphism $\Psi((\Omega_+, \omega)) : \Psi(\mathcal{P}') := G_{\mathcal{P}'} \rightarrow G_{\mathcal{P}''} =: \Psi(\mathcal{P}'')$ is defined by

$$\Psi((\Omega_+, \omega))_A : \quad g'_+ \mapsto \Omega_+(g'_+) \quad , \quad (1 + \eta Y') \mapsto (1 + \eta \omega(Y')) \quad (5.13)$$

for all $A \in (\mathbf{Wsalg})_{\mathbb{K}}$, $g'_+ \in G'_+(A)$, $\eta \in A_1$, $Y' \in \mathfrak{g}'$, with $\mathcal{P}' = (G'_+, \mathfrak{g}')$ and $\mathcal{P}'' = (G''_+, \mathfrak{g}'')$.

Proof. What is still left to prove is only that the given definition for $\Psi((\Omega_+, \omega))$ actually makes sense, as all the rest is already proved by our previous analysis — in particular, by §5.2.6 above — or is elementary. Now, (5.13) above fixes the values of our would-be morphism $\Psi((\Omega_+, \omega))_A$ on generators of $\Psi(\mathcal{P}')(A) := G_{\mathcal{P}'}(A)$: then a straightforward check shows that all defining relations among such generators — inside $G_{\mathcal{P}'}(A)$ — are mapped to corresponding (defining) relations in $G_{\mathcal{P}''}(A)$, thus providing a unique, well-defined group morphism as required. However, we must show that *this is a morphism of \mathcal{A}_0 -manifolds too*, which needs some extra work.

Let $\{Y'_i\}_{i \in I}$ and $\{Y''_j\}_{j \in J}$ be \mathbb{K} -bases of \mathfrak{g}'_1 and \mathfrak{g}''_1 respectively, both endowed with some fixed total order. Accordingly, both $G_{\mathcal{P}'}(A)$ and $G_{\mathcal{P}''}(A)$ admit factorizations as in Proposition 5.2.5(a) — say of type $G_+ \times G_{\leq}$. In particular, any given $g' \in G_{\mathcal{P}'}(A)$ uniquely factors into $g' = g'_+ \cdot \prod_{i \in I} (1 + \eta_i Y'_i)$; then $\Psi((\Omega_+, \omega))_A$, being a group morphism, maps g' onto

$$\Psi((\Omega_+, \omega))_A(g') = \Omega_+(g'_+) \cdot \prod_{i \in I} (1 + \eta_i \omega(Y'_i))$$

and from this, letting $\omega(Y'_i) = \sum_{j \in J} c_{i,j} Z_j$ — with $c_{i,j} \in \mathbb{K}$ — we get

$$\Psi((\Omega_+, \omega))_A(g') = \Omega_+(g'_+) \cdot \overrightarrow{\prod}_{i \in I} \left(1 + \eta_i \left(\sum_{j \in J} c_{i,j} Z_j\right)\right) = \Omega_+(g'_+) \cdot \overrightarrow{\prod}_{i \in I} \prod_{j \in J} \left(1 + \eta_i c_{i,j} Z_j\right) \quad (5.14)$$

where in the second product in the rightmost term the order of factors is irrelevant, as they do commute with each other. Now we must re-order the result according to the factorization of $G_{\mathcal{P}''}(A)$ of the form $G_+ \times G_-^{\leq}$; in doing this, when we reorder the second factor $\overrightarrow{\prod}_{i \in I} \prod_{j \in J} (1 + \eta_i c_{i,j} Z_j)$ in (5.14) above we find — via calculations as in the proof of Proposition 5.2.1 (which means like those for Proposition 4.2.6(b)) — an outcome of the form $\prod_{r=1}^n (1 + a_r X_r)_{G_+''} \cdot \overrightarrow{\prod}_{j \in J} (1 + \alpha_j Z_j)$ where

- (a) the X_r 's belong to \mathfrak{g}_0'' ,
- (b) the a_r 's are (even) polynomial expressions in the η_i 's,
- (c) the α_j 's are (odd) polynomial expressions in the η_i 's,

Overall, this implies that the map

$$\overrightarrow{\prod}_{i \in I} (1 + \eta_i Y'_i) \mapsto \prod_{r=1}^n (1 + a_r X_r)_{G_+''} \cdot \overrightarrow{\prod}_{j \in J} (1 + \alpha_j Z_j)$$

is a map of \mathcal{A}_0 -manifolds from $(G_{\mathcal{P}'}_-)^{\leq}(A)$ to $G_{\mathcal{P}''}(A)$. But $\Omega_+ : G'_+(A) \longrightarrow G''_+(A)$ is a map of \mathcal{A}_0 -manifolds too, by assumptions; this along with the previous remark and (5.14) above eventually implies that $\Psi((\Omega_+, \omega))_A$ is a map of \mathcal{A}_0 -manifolds as claimed. \square

6 A new equivalence $(\mathbf{sHCp}) \cong (\mathbf{Lsgrp})$.

In Section 5 we introduced a functor, denoted Ψ , from \mathbf{sHCp} 's — of any type: real smooth, real analytic or complex holomorphic — to Lie supergroups — of the same type; in particular, this goes the other way round with respect to the “natural” functor Φ from Lie supergroups to \mathbf{sHCp} 's considered in Section 3. In the present section we shall now show that these two functors are quasi-inverse to each other, so that they provide equivalences between the category of super Harish-Chandra pairs (of a given type) and the category of Lie supergroups (of the same type).

6.1 The functor Ψ as a quasi-inverse to Φ : proof of $\Phi \circ \Psi \cong id_{(\mathbf{sHCp})}$

In this subsection we cope with the first half of our task, namely proving that $\Phi \circ \Psi \cong id_{(\mathbf{sHCp})}$; indeed, this amounts to be the easy part of the job.

Proposition 6.1.1.

Given functors Φ as in Theorem 3.2.6 and Ψ as in Theorem 5.2.7, we have $\Phi \circ \Psi \cong id_{(\mathbf{sHCp})}$ — where “ (\mathbf{sHCp}) ” can/must be read as either $(\mathbf{sHCp})_{\mathbb{R}}^{\infty}$, or $(\mathbf{sHCp})_{\mathbb{R}}^{\omega}$, or $(\mathbf{sHCp})_{\mathbb{C}}^{\omega}$, and Φ and Ψ must be taken as working onto the corresponding types of \mathbf{sHCp} 's or Lie supergroups.

Proof. This follows almost directly from definitions. Indeed, let us consider a super Harish-Chandra pair (of either smooth, or analytic, or holomorphic type) $\mathcal{P} := (G_+, \mathfrak{g})$ to start with, and let $G_{\mathcal{P}} = \Psi(\mathcal{P})$, so that $(\Phi \circ \Psi)(\mathcal{P}) = \Phi(G_{\mathcal{P}}) = ((G_{\mathcal{P}})_0, \text{Lie}(G_{\mathcal{P}}))$. Then, by the very construction of $G_{\mathcal{P}}$ we have $(G_{\mathcal{P}})_0 = G_+$. In addition, following the very definition of $\text{Lie}(G)$ given in Definition

3.2.3 and exploiting the factorization $G_{\mathcal{P}} \cong G_+ \times G_-^<$ from Proposition 5.2.5, one finds — by straightforward, bare hands computation, — that

$$\mathrm{Lie}(G_{\mathcal{P}}) = \mathrm{Lie}(G_+ \times G_-^<) = \mathrm{Lie}(G_+) \oplus T_e(G_-^<) = \mathcal{L}_{\mathfrak{g}_0} \oplus \mathcal{L}_{\mathfrak{g}_1} = \mathcal{L}_{\mathfrak{g}}$$

(cf. §2.2.3 and Proposition 3.2.4 for notation), which means — identifying $\mathcal{L}_{\mathfrak{g}}$ with \mathfrak{g} as usual — simply $\mathrm{Lie}(G_{\mathcal{P}}) = \mathfrak{g}$, this being an identification as Lie \mathbb{K} -superalgebras. Therefore, in the end

$$(\Phi \circ \Psi)(\mathcal{P}) = \Phi(G_{\mathcal{P}}) = ((G_{\mathcal{P}})_0, \mathrm{Lie}(G_{\mathcal{P}})) \cong (G_+, \mathfrak{g}) = \mathcal{P}$$

which means that $\Phi \circ \Psi$ acts on objects — up to natural isomorphisms — as the identity, q.e.d.

As to morphisms, let $(\Omega_+, \omega) : \mathcal{P}' = (G'_+, \mathfrak{g}') \longrightarrow (G''_+, \mathfrak{g}'') = \mathcal{P}''$ be a morphism of super Harish-Chandra pairs and $\varpi := \Psi((\Omega_+, \omega)) : \Psi(\mathcal{P}') = G_{\mathcal{P}'} \longrightarrow G_{\mathcal{P}''} = \Psi(\mathcal{P}'')$ the corresponding (via Ψ) morphism between supergroups; we aim to prove that $(\Phi \circ \Psi)((\Omega_+, \omega)) = \Phi(\varpi)$ actually coincides — up to the natural isomorphisms $(\Phi \circ \Psi)(\mathcal{P}') \cong \mathcal{P}'$ and $(\Phi \circ \Psi)(\mathcal{P}'') \cong \mathcal{P}''$ mentioned above — with (Ω_+, ω) itself.

By definition $\Phi(\varpi) := (\varpi|_{(G_{\mathcal{P}'})_0}, d\varpi)$. Now, on the one hand by the very construction of ϖ we have $\varpi|_{(G_{\mathcal{P}'})_0} = \Psi((\Omega_+, \omega))|_{G'_+} = \Omega_+$. On the other hand, like in the proof of Theorem 5.2.7 we consider factorizations $G_{\mathcal{P}'} = G'_+ \times G'^{<}_-$ and $G_{\mathcal{P}''} = G''_+ \times G''^{<}_-$; using these and the very constructions we find that the action of $d\varpi$ onto $T_e(G_{\mathcal{P}'}) = T_e(G'_+) \oplus T_e(G'^{<}_-) = \mathfrak{g}_0 \oplus \mathfrak{g}_1 = \mathfrak{g}$ is given by $d\varpi|_{\mathfrak{g}_0} = d\Omega_+ = \omega|_{\mathfrak{g}_0}$ onto \mathfrak{g}_0 and by $d\varpi|_{\mathfrak{g}_1} = \omega|_{\mathfrak{g}_1}$ onto \mathfrak{g}_1 ; eventually, this gives $d\varpi = d\varpi|_{\mathfrak{g}_0} \oplus d\varpi|_{\mathfrak{g}_1} = \omega_0 \oplus \omega_1 = \omega$ as expected, so that $\Phi(\varpi) := (\varpi|_{(G_{\mathcal{P}'})_0}, d\varpi) = (\Omega_+, \omega)$. \square

6.2 The functor Ψ as a quasi-inverse to Φ : proof of $\Psi \circ \Phi \cong \mathrm{id}_{(\mathbf{Lsgrp})}$

We can now add up the last missing piece, namely proving that the composition $\Psi \circ \Phi$ is isomorphic to the identity functor on Lie supergroups. This last step, together with Proposition 6.1.1, will prove that the functors Φ and Ψ are quasi-inverse to each other, in particular they are equivalences between Lie supergroups and super Harish-Chandra pairs (and viceversa). Formally, the statement takes the following form:

Proposition 6.2.1.

Given functors Φ as in Theorem 3.2.6 and Ψ as in Theorem 5.2.7, we have $\Psi \circ \Phi \cong \mathrm{id}_{(\mathbf{Lsgrp})}$ — where “ (\mathbf{Lsgrp}) ” can/must be read as either $(\mathbf{Lsgrp})_{\mathbb{R}}^{\infty}$, or $(\mathbf{Lsgrp})_{\mathbb{R}}^{\omega}$, or $(\mathbf{Lsgrp})_{\mathbb{C}}^{\omega}$, and Φ and Ψ must be taken as working onto the corresponding types of Lie supergroups or $s\mathrm{HCp}$ ’s.

Proof. Given a Lie supergroup G , set $\mathfrak{g} := \mathrm{Lie}(G)$ and $\mathcal{P} := \Phi_g(G) = (G_0, \mathfrak{g})$. We look at the supergroup $\Psi(\Phi(G)) = \Psi(\mathcal{P}) := G_{\mathcal{P}}$, aiming to prove that the latter is naturally isomorphic to G .

Given $A \in (\mathbf{salg})_{\mathbb{K}}$, by abuse of notation we denote with the same symbol any element $g_0 \in G_0(A)$ as belonging to $G(A)$ — via the embedding of $G_0(A)$ into $G(A)$ — and as an element of $G_{\mathcal{P}}(A)$ — actually, one of the distinguished generators of $G_{\mathcal{P}}(A)$ given from scratch.

With this convention, it is immediate to see that Lemma 4.2.4 yields the following: *there exists a unique group morphism $\Omega_A : G_{\mathcal{P}}(A) \longrightarrow G(A)$ such that $\Omega_A(g_0) = g_0$ for all $g_0 \in G_0(A)$ and $\Omega_A((1 + \eta_i Y_i)) = (1 + \eta_i Y_i)$ for all $\eta_i \in A_1$, $i \in I$.* Indeed, thanks to that lemma we know that the defining relations among generators of $G_{\mathcal{P}}(A)$ are also satisfied by their images in $G(A)$ through Ω_A under the above prescription.

Due to the factorization (4.12) in Proposition 4.2.6, we have also that *the morphism Ω_A is actually surjective.* Even more, the *Global Splitting Theorem* for G (namely, Theorem 4.2.9) and

for $G_{\mathcal{P}}$ (that is, Proposition 5.2.5) together easily imply that *the morphism Ω_A is also injective*, hence *it is a group isomorphism*. Finally, it is clear that all these Ω_A 's are natural in A , thus altogether they provide an isomorphism between $G_{\mathcal{P}} = \Psi_g(\Phi_g(G))$ and G , which ends the proof. \square

7 Linear case and representations.

In this subsection we spend, somewhat shortly, a few words about the fallout of the existence of a category equivalence between Lie supergroups and super Harish-Chandra pairs, in particular the one that we realize via the functor Ψ presented in §5.1, in representation theory.

7.1 The linear case

The very construction of our functor Ψ — and of Φ as well — gains a much more concrete meaning when the super Harish-Chandra pairs and Lie supergroups we deal with are *linear*.

Indeed, assume first that the Lie supergroup G is linear, i.e. it embeds into some $\mathrm{GL}(V)$, where V is a suitable superspace (in other words, there exists a faithful G -module V). Then both G_0 and $\mathfrak{g} := \mathrm{Lie}(G)$ embed into $\mathrm{End}(V)$, and the relations linking them — that we formalize saying that “ (G_0, \mathfrak{g}) is a super Harish-Chandra pair” — actually are relations among elements of the unital, associative superalgebra $\mathrm{End}(V)$. Conversely, basing on this, one can formally make up the notion of “linear super Harish-Chandra pair” as being any $\mathrm{sHCp}(G_0, \mathfrak{g})$ such that both G_0 and \mathfrak{g} embed into some $\mathrm{End}(V)$, and the compatibility relations linking G_0 and \mathfrak{g} actually hold true as relations inside the superalgebra $\mathrm{End}(V)$ itself — cf. [15], Definition 4.2.1(b) for such a formalization in the setup of algebraic supergroups and sHCp 's (it easily adapts to the present Lie setup). The above then tells us, in short, that if G is linear then its associated super Harish-Chandra pair $\Phi(G) =: \mathcal{P}$ is linear too — both being linearized through their faithful representation onto V .

On the other hand, let us start now with a linear sHCp , say $\mathcal{P} = (G_+, \mathfrak{g})$: so the latter is embedded (in the obvious sense) into the $\mathrm{sHCp}(\mathrm{GL}_0(V), \mathfrak{gl}(V))$ for some representation superspace V . Thus for any $A \in (\mathbf{Wsalg})_{\mathbb{K}}$ both $G_+(A)$ and $A \otimes_{\mathbb{K}} \mathfrak{g}$ are embedded into $(\mathrm{End}(V))(A)$, with relations among them — inside the algebra $(\mathrm{End}(V))(A)$ — induced by the very notion of linear sHCp . Now, one can also consider inside $(\mathrm{End}(V))(A)$ all elements of the form $\exp(\eta_i Y_i) = (1 + \eta_i Y_i)$ — with $\eta_i \in A_1$, for any fixed, totally ordered \mathbb{K} -basis $\{Y_i\}_{i \in I}$ of \mathfrak{g}_1 as usual — that actually belong to $(\mathrm{GL}(V))(A)$: similarly, we clearly have $G_+(A) \subseteq (\mathrm{GL}(V))(A)$ too. Therefore, *one can take inside $(\mathrm{GL}(V))(A)$ the subgroup $G_{\mathcal{P}}^V(A)$ generated by $G_+(A)$ and by all the $(1 + \eta_i Y_i)$'s.*

A trivial check shows that the elements from $G_+(A)$ and the $(1 + \eta_i Y_i)$'s enjoy all relations that enter in the very definition $G_{\mathcal{P}}(A)$ for their parallel counterparts: thus, *there exists a (unique) group morphism $\omega_A : G_{\mathcal{P}}(A) \longrightarrow G_{\mathcal{P}}^V(A)$ such that $\omega_A(g_+) = g_+$ and $\omega_A((1 + \eta_i Y_i)) = (1 + \eta_i Y_i)$ for all $g_+ \in G_+(A)$, $\eta_i \in A_1$, $i \in I$; in addition, by construction this ω_A is clearly onto.*

On the other hand, $G_{\mathcal{P}}^V(A)$ acts faithfully on V , by definition: indeed, it is linear, namely “linearized by V ”. A key fact then is that this linearization allows one to show that *the like of the Global Splitting Theorem* — cf. Theorem 4.2.9 and Proposition 5.2.5 — *does hold true for $G_{\mathcal{P}}^V(A)$* ; indeed, one can apply the same analysis and arguments used in the proofs of either Theorem 4.2.9 or Proposition 5.2.5, *but* for one single change: the given linearization of $G_{\mathcal{P}}^V(A)$ (induced by the initial linearization of the sHCp \mathcal{P} we started with) has to replace the following key ingredient:

- in the proof of Theorem 4.2.9, one has that $G(A) := \coprod_{x \in |G|} \mathrm{Hom}(\mathbf{salg}_{\mathbb{K}}(\mathcal{O}_{|G|,x}, A)$,
- in the proof of Proposition 5.2.5 (and of the lemmas before it, mostly), one has that $G_{\mathcal{P}}(A)$ is acting onto $V := \mathrm{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}}(\mathbf{1})$ — see (5.7).

In fact, in both cases — of either $G(A)$ or $G_{\mathcal{P}}(A)$ — the group under exam is realized as a group of maps (linear operators, in the second case), and these are rich enough to “separate (enough) points” so to guarantee the uniqueness of factorization(s) that is the core part of the global splitting theorem. In the case of $G_{\mathcal{P}}^V(A)$ instead, its built-in linearization provides a similar realization as “group of maps”, and this again allows to separate enough points to get global splitting(s) for $G_{\mathcal{P}}^V(A)$ too. Finally, in force of the global splitting theorem for $G_{\mathcal{P}}(A)$ and for $G_{\mathcal{P}}^V(A)$ one can apply again the arguments used in the proof of Proposition 6.2.1 and successfully prove that *the above group (epi)morphism $\omega_A : G_{\mathcal{P}}(A) \longrightarrow G_{\mathcal{P}}^V(A)$ is also injective, hence it is an isomorphism.*

By construction all these isomorphisms ω_A are natural in A , hence they give altogether a functor isomorphism $\omega : G_{\mathcal{P}} \xrightarrow{\cong} G_{\mathcal{P}}^V$. Therefore $G_{\mathcal{P}} \cong G_{\mathcal{P}}^V$, which means that we found a different, concrete realization of $G_{\mathcal{P}}$, that is now constructed explicitly as the *linear* Lie supergroup $G_{\mathcal{P}}^V$.

7.2 Representations 1: supergroup modules vs. sHCp modules

An important byproduct of the equivalence between Lie supergroups and super Harish-Chandra pairs comes as application to representation theory. Indeed, let G and \mathcal{P} respectively be a supergroup and a super Harish-Chandra pair — of either type (smooth, etc.) as usual — over \mathbb{K} that correspond to each other through the equivalence presented in §6 — namely, $G = \Psi(\mathcal{P})$ and $\mathcal{P} = \Phi(G)$. Then we let $G\text{-Mod}$ and $\mathcal{P}\text{-Mod}$ respectively be the category of G -modules and of \mathcal{P} -modules (again of type smooth, etc.); in short, here we mean that a G -module is the datum of a finite dimensional supermodule M' with a morphism $\Omega : G \longrightarrow \mathrm{GL}(M')$ of Lie supergroups (in the proper category), whereas a \mathcal{P} -module is the datum of a finite dimensional supermodule M'' with a morphism $(\Omega_+, \omega) : \mathcal{P} \longrightarrow (\mathrm{GL}(M'')_0, \mathfrak{gl}(M''))$ of super Harish-Chandra pairs.

Just to fix notation, we assume to be in the real smooth case, the other cases being entirely similar. Let's assume M' is a G -module; applying $\Phi : (\mathbf{Lsgrp})_{\mathbb{R}}^{\infty} \longrightarrow (\mathbf{sHCp})_{\mathbb{R}}$ to the morphism $\Omega : G \longrightarrow \mathrm{GL}(M')$ we find a morphism $\Phi(\Omega) : \Phi(G) \longrightarrow \Phi(\mathrm{GL}(M'))$ between the corresponding objects in $(\mathbf{sHCp})_{\mathbb{R}}$. But $\Phi(G) = \mathcal{P}$ by assumption and $\Phi(\mathrm{GL}(M')) = (\mathrm{GL}(M')_0, \mathfrak{gl}(M'))$, so what we have is a morphism $\Phi(\Omega) : \mathcal{P} \longrightarrow (\mathrm{GL}(M')_0, \mathfrak{gl}(M'))$ making M' into a \mathcal{P} -module.

Conversely, let M'' be a \mathcal{P} -module. Applying the functor $\Psi : (\mathbf{sHCp}) \longrightarrow (\mathbf{Lsgrp})_{\mathbb{R}}^{\infty}$ to the corresponding morphism $(\Omega_+, \omega) : \mathcal{P} \longrightarrow (\mathrm{GL}(M'')_0, \mathfrak{gl}(M''))$ we get a morphism between the corresponding supergroups, namely $\Psi((\Omega_+, \omega)) : \Psi(\mathcal{P}) \longrightarrow \Psi((\mathrm{GL}(M'')_0, \mathfrak{gl}(M'')))$. As $\Psi(\mathcal{P}) = G$ by and $\Psi((\mathrm{GL}(M'')_0, \mathfrak{gl}(M''))) = \mathrm{GL}(M'')$, we find a morphism $\Psi((\Omega_+, \omega)) : G \longrightarrow \mathrm{GL}(M'')$ in $(\mathbf{Lsgrp})_{\mathbb{R}}^{\infty}$ which makes M'' into a G -module. In fact, in this way the G -action that one gets on M'' is just what one obtains by direct application of the recipe in §7.1 to the *linear* super Harish-Chandra pair $(\Omega_+, \omega)(\mathcal{P})$ inside $(\mathrm{GL}(M'')_0, \mathfrak{gl}(M''))$ that is the image of \mathcal{P} through (Ω_+, ω) .

The reader can easily check that the previous discussion — extended to the real analytic case and to the complex holomorphic case as well — has the following outcome:

Theorem 7.2.1. *Let G and \mathcal{P} be a Lie supergroup and a super Harish-Chandra pair (of smooth, analytic or holomorphic type) over \mathbb{K} corresponding to each other as above. Then:*

- (a) *for any fixed finite dimensional \mathbb{K} -supermodule M , the above constructions provide two bijections, inverse to each other, between G -module structures and \mathcal{P} -module structures on M ;*
- (b) *the whole construction above is natural in M , in that the above bijections between G -module structures and \mathcal{P} -module structures over two finite dimensional \mathbb{K} -supermodules \widehat{M} and \widetilde{M} are compatible with \mathbb{K} -supermodule morphisms from \widehat{M} to \widetilde{M} . Thus, all the bijections mentioned in (a), for all different M 's, do provide equivalences, quasi-inverse to each other, between the category of all (finite dimensional) G -modules and the category of all (finite dimensional) \mathcal{P} -modules.*

7.3 Representations 2: induction from G_0 to G

Let G be a supergroup (of any type), with associated classical subsupergroup G_0 . Let V be any G_0 -module: we shall now present an explicit construction of the *induced G -module* $Ind_{G_0}^G(V)$.

Being a G_0 -module, V is also, automatically, a \mathfrak{g}_0 -module. Then one does have the induced \mathfrak{g} -module $Ind_{\mathfrak{g}_0}^{\mathfrak{g}}(V)$, which can be realized as

$$Ind_{\mathfrak{g}_0}^{\mathfrak{g}}(V) = Ind_{U(\mathfrak{g}_0)}^{U(\mathfrak{g})}(V) = U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_0)} V$$

By construction, it is clear that this bears also a unique structure of G_0 -module which is compatible with the \mathfrak{g} -action and coincides with the original G_0 -action on $\mathbb{K} \otimes_{U(\mathfrak{g}_0)} V \cong V$ given from scratch. Indeed, we can describe explicitly this G_0 -action, as follows. First, by construction we have

$$Ind_{\mathfrak{g}_0}^{\mathfrak{g}}(V) = U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_0)} V = \bigwedge \mathfrak{g}_1 \otimes_{\mathbb{K}} V$$

— because $U(\mathfrak{g}) \cong \bigwedge \mathfrak{g}_1 \otimes_{\mathbb{K}} U(\mathfrak{g}_0)$ as a \mathbb{K} -module, by the PBW theorem for Lie superalgebras, see (5.6) — with the \mathfrak{g}_0 -action given by $x.(y \otimes v) = ad(x)(y) \otimes v + y \otimes (x.v)$ for $x \in \mathfrak{g}_0$, $y \in \bigwedge \mathfrak{g}_1$, $v \in V$, where by ad we denote the unique \mathfrak{g}_0 -action on $\bigwedge \mathfrak{g}_1$ by algebra derivations induced by the adjoint \mathfrak{g}_0 -action on \mathfrak{g}_1 . Second, this action clearly integrates to a (unique) G_0 -action given by $g_0.(y \otimes v) := Ad(g_0)(y) \otimes (g_0.v)$ for $g_0 \in G_0$, $y \in \bigwedge \mathfrak{g}_1$, $v \in V$, where we write Ad for the unique G_0 -action on $\bigwedge \mathfrak{g}_1$ by algebra automorphisms induced by the adjoint G_0 -action on \mathfrak{g}_1 .

The key point is that the above G_0 -action and the built-in \mathfrak{g} -action on $Ind_{\mathfrak{g}_0}^{\mathfrak{g}}(V)$ are actually compatible, in the sense that they make $Ind_{\mathfrak{g}_0}^{\mathfrak{g}}(V)$ into a (G_0, \mathfrak{g}) -module, i.e. a module for the super Harish-Chandra pair $\mathcal{P} := (G_0, \mathfrak{g}) = \Phi(G)$. Since $\Psi((G_0, \mathfrak{g})) = G$, by §7.2 we have that $Ind_{\mathfrak{g}_0}^{\mathfrak{g}}(V)$ bears a unique structure of G -module which correspond to the previous \mathcal{P} -action — i.e., it yields (by restriction and “differentiation”) the previously found G_0 -action and \mathfrak{g} -action. Explicitly, in down-to-earth terms what happens is the following. The action of the sHCp $\mathcal{P} := (G_0, \mathfrak{g}) = \Phi(G)$ on $W := Ind_{\mathfrak{g}_0}^{\mathfrak{g}}(V)$ is given by the G_0 -action (induced by the original action on V) and a compatible \mathfrak{g} -action: in particular, the latter defines an action of each element Y_i of any fixed, totally ordered basis of \mathfrak{g}_1 . Thus each Y_i acts as an operator in $End(W)$, hence — when choosing odd coefficients η_i in any $A \in (\mathbf{Wsalg})_{\mathbb{K}}$, etc. — we also have unique, well-defined corresponding operators $(1 + \eta_i Y_i)$ in $(End(W))(A)$ that actually belong to $(GL(W))(A)$ indeed. It so happens (trivial check) that these $(1 + \eta_i Y_i)$ ’s altogether enjoy among themselves and with the operators given by the G_0 -action the very relations that enter in the definition of $\Psi(\mathcal{P}) = G_{\mathcal{P}}$; thus we eventually have a well-defined action of $\Psi(\mathcal{P})$ on W , extending the initial one by G_0 : but $\Psi(\mathcal{P}) = G$, hence we are done.

Therefore, we define as $Ind_{G_0}^G(V)$ the space $W := Ind_{\mathfrak{g}_0}^{\mathfrak{g}}(V)$ endowed with this G -action: one easily checks that this construction is functorial in V and has the universal property which makes it into the adjoint of “restriction” (from G -modules to G_0 -modules), so it has all rights to be called “induction” functor (from G_0 -modules to G -modules).

In addition, if the original G_0 -module V is faithful then the induced G -module $Ind_{G_0}^G(V)$ is faithful too: in particular, this means that if G_0 is linearizable, then G is linearizable too; more precisely, from a linearization of G_0 one can construct (via induction) a linearization of G as well.

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